The limit of discounted utilitarianism

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This paper presents an infinite-horizon version of intergenerational utilitarianism. By studying discounted utilitarianism as the discount factor tends to one, we obtain a new welfare criterion: limit-discounted utilitarianism (LDU). We show that LDU meets standard assumptions on efficiency, equity, and interpersonal comparability, but allows us to compare more pairs of utility streams than commonly used utilitarian criteria do, including the overtaking criterion and the catching-up criterion. We also introduce a principle of compensation for postponements of utility streams and use it to characterize the LDU criterion on a restricted domain.

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JEL classification. D63, D70, D90.

1. Introduction

Utilitarianism is the normative theory which says that the best social policy in a set of alternatives is the one with the greatest total welfare. Total welfare is often defined as the sum of the utilities for all members of society. This notion of maximizing aggregate utility becomes problematic in infinite-horizon models. The problem of aggregating infinite utility streams $u = (u_1, u_2, \ldots)$, representing the utilities of present and future generations, has occupied philosophers and economists for over a century. Discounted utilitarianism provides a popular criterion for evaluating such streams. But since discounting assigns smaller weight to future generations, discounted utilitarianism has also been the subject of heavy criticism.[1] For instance, Ramsey (1928, p. 543) calls discounting the utility of future generations an "ethically indefensible" practice that "arises merely from the weakness of the imagination". Koopmans (1960), in his axiomatic approach to discounted utilitarianism, formally defines time preference through the concept of *impatience*. Subsequently, a long tradition in welfare economics studies social

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[1] This criticism goes back at least to Sidgwick (1907) and Pigou (1920).

preferences that combine two properties: the Strong Pareto axiom, saying that Pareto improvements lead to better outcomes, and Anonymity, which models the equal treatment of any pair of generations by insisting that preferences are not affected by permuting the utility level of two generations. The formulation of criteria that satisfy these two properties, however, is hampered by two major impossibility theorems. Firstly, if complete and transitive preferences over infinite utility streams satisfy Strong Pareto and Anonymity, they cannot be represented by a social welfare function.[2] Secondly, such preferences cannot be obtained by constructive methods.[3] Informally, all explicit descriptions of transitive preferences satisfying Strong Pareto and Anonymity are incomplete.

The literature on explicitly defined intergenerational preferences satisfying Strong Pareto and Anonymity therefore uses incomplete preferences. A social welfare relation (SWR) is a binary relation that is reflexive and transitive, but not necessarily complete. Utilitarian SWRs often address the problem of ordering infinite-horizon streams u and v by looking instead at their partial sums over a long, but finite horizon and seeing what happens if this horizon diverges to infinity.[4] This is the case, for instance, for the overtaking criterion (von Weizsäcker, 1965), the catching-up criterion (Gale, 1967), and the utilitarian SWR of Basu and Mitra (2007).

Our approach is different. Instead of comparing u and v on the basis of the limit behavior of the partial sums of $u - v$, we look at its discounted sum and let the discount factor tend to one.[5] Our *limit-discounted utilitarian* (LDU) criterion declares u to be at least as good as v if the (lower) limit is nonnegative. This also avoids the critique of Ramsey (1928) because in the limit, each pair of generations is treated equally. Our main results include:

The Compensation Principle. Informally, the Compensation Principle says that a utility stream can be postponed for one generation if the first generation is compensated by the average utility over all generations. The precise formulation is in Section 2 where we also use common axioms to motivate why this average is a reasonable compensation. Limit-discounted utilitarianism satisfies the Compensation Principle and this property — or more generally, its ability to compare well-behaved streams with compensated postponements — enables us to rank pairs of streams when other SWRs can't; see Example 2 for a concrete example.

Our criterion, like overtaking and catching-up, ranks the periodic stream $u = (1, 0, 1, 0, \dots)$ above $v = (0, 1, 0, 1, \dots)$: u has average $1/2$, so the Compensation Principle says that u is equivalent with $w = (1/2, 1, 0, 1, 0, \dots)$ where u is postponed for one generation and the first receives compensation $1/2$. And w Pareto dominates v, so by Strong Pareto and transitivity, u is preferred to v. Now, social welfare relations that satisfy Anonymity cannot be impatient in the traditional sense of Koopmans (1960).[6] But some authors[7] argue that strictly preferring u to v is a sign of a different type of impatience and advocate stronger anonymity notions using classes of infinite permutations that require u and v to be equivalent. This equivalence, how-

[2] Diamond's (1965) version of this result was obtained under an additional continuity assumption on the social welfare function. The general impossibility theorem, without this assumption, is due to Basu and Mitra (2003).

[3] The existence of complete, transitive binary relations satisfying Strong Pareto and Anonymity was established by Svensson (1980). Zame (2007, Theorem 4) and Lauwers (2010, p. 33) show that Svensson's existence theorem cannot be proved without using the Axiom of Choice.

[4] Formally, $\liminf_{T\to\infty} \sum_{t=1}^{T} (u_t - v_t) \geq 0$ must hold if $u = (u_1, u_2, \dots)$ is at least as good as $v = (v_1, v_2, \dots)$.

[5] Discounting with discount factors tending to one has been used extensively in the literature on stochastic games and dynamic optimization; see, e.g., Liggett and Lippman (1969), Lippman (1969), Dutta (1991), Sennott (1999), and Bishop et al. (2014). Basu and Mitra (2007, p. 360-361) defend the relevance of vanishing discount rates for intergenerational equity in a "robustness check" of their welfare criterion. They attribute the idea behind the robustness check to Jörgen Weibull. Limit-discounted utilitarianism can be seen as a concretization of their line of thought.

[6] The definition of Koopmans (1960, p. 296) is for social welfare relations defined by social welfare functions. By Banerjee and Mitra's (2007, Sec. 2.2.2) more general ordinal formulation, impatience is displayed by strictly preferring a stream u with $u_s > u_t$ for some $s < t$ to the stream obtained by switching u_s and u_t. With Anonymity, these two streams are equivalent.

[7] For example, Lauwers (1995), Fleurbaey and Michel (2003), and Heal (2005).

ever, implies a violation of Strong Pareto or Koopmans's (1960) Stationarity axiom, two often combined conditions in intertemporal contexts.[8] The Compensation Principle addresses an aspect of time preference — compensating for postponements — in a way that is not in conflict with these conditions.

A characterization. Theorem 1 shows that Strong Pareto, the Compensation Principle, and Additivity (a standard translation invariance axiom) characterize limit-discounted utilitarianism on the set of pairs of utility streams with a summable or eventually periodic difference.

Basic properties. Theorem 2 shows that limit-discounted utilitarianism satisfies standard assumptions on efficiency, equity, and interpersonal comparability. It also has a continuity property relating preferences over infinite streams with long, finite-horizon truncations. This continuity requirement is a less demanding version of a similar requirement in Brock's (1970) classical characterization of the overtaking criterion. Moreover, it satisfies the intuitive utilitarian requirement that summable streams with a larger sum are strictly preferred to those with a smaller sum; alternatives with equal finite sums are equivalent.

Comparison to other utilitarian criteria. Theorem 3 compares our criterion to overtaking, catching-up, and the Basu-Mitra criterion. Briefly: if u is weakly preferred to v according to any of these three criteria, then the same is true for limit-discounted utilitarianism. For the Basu-Mitra criterion, the implication holds for strict preference as well. Moreover, Table 1 summarizes similarities and differences at an axiomatic level. Throughout the paper we also indicate well-behaved pairs of streams that can be compared using limit-discounted utilitarianism, but not always under any of these other criteria; these include streams and their compensated postponements and pairs of streams whose difference is eventually periodic. Finally, limit-discounted utilitarianism is closely related to Abel's summation method from the theory of divergent series and our generalization of the classical theorem of Frobenius (1880) in Lemma 1 provides sufficient conditions for streams to be comparable using our criterion.

2. Limit-discounted utilitarianism

The purpose of this section is threefold. We define limit-discounted utilitarianism in (1). Theorem 1 characterizes its behavior on a domain that is sufficiently large to include the most commonly discussed examples, and Theorem 2 shows that it satisfies standard desiderata on, for instance, efficiency, equity, and interpersonal comparability.

The following notation will be used: $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers, $\mathbb{R}$ the set of real numbers,

$$\mathscr{U} = \{u \in \mathbb{R}^{\mathbb{N}} : \sup_{t \in \mathbb{N}} |u_t| < +\infty\}$$

the set of bounded utility streams $u = (u_1, u_2, \ldots)$, where u_t denotes the welfare level of generation $t \in \mathbb{N}$. A *social welfare relation* (SWR) is a reflexive and transitive binary relation $\succsim$ on $\mathscr{U}$. For $u, v \in \mathscr{U}$, $u \succsim v$ means that society considers u to be at least as good as v, $u \sim v$ means that $u \succsim v$ and $v \succsim u$, whereas $u \succ v$ means that $u \succsim v$ but not $v \succsim u$.

Limit-discounted utilitarianism compares streams $u \in \mathscr{U}$ using their discounted sum

$$\sigma_\delta(u) = \sum_{t=1}^{\infty} \delta^{t-1} u_t,$$

[8] If a SWR satisfies Strong Pareto and Stationarity, $u = (1, 0, 1, 0, \ldots)$ and $v = (0, 1, 0, 1, \ldots)$ are incomparable or u is strictly preferred to v. If, to the contrary, $(0, 1, 0, 1, \ldots)$ is at least as good as $(1, 0, 1, 0, \ldots)$, then Stationarity — appending an identical first coordinate — implies that $(1, 0, 1, 0, 1, \ldots)$ is at least as good as $(1, 1, 0, 1, 0, \ldots)$. By transitivity, $(0, 1, 0, 1, \ldots)$ is at least as good as $(1, 1, 0, 1, 0, \ldots)$. But that contradicts Strong Pareto. For results along these lines, see Dutta (2008), Asheim et al. (2010), and Asheim and Banerjee (2010).

but lets the discount factor $\delta \in (0,1)$ tend to one to give equal weight to each pair of generations. Formally:[9]

DEFINITION 1. *Limit-discounted utilitarianism* (LDU) is the binary relation $\succsim_{\text{LDU}}$ defined, for all $u, v \in \mathscr{U}$, by

$$u \succsim_{\text{LDU}} v \iff \liminf_{\delta \to 1^-} \sigma_\delta(u - v) \geq 0. \tag{1}$$

In the literature on divergent series, the lower limit in (1) is called the lower Abel sum and the limit $\lim_{\delta \to 1^-} \sigma_\delta(u - v)$, if it exists, is called the Abel-sum of the series $\sum_{t=1}^{\infty}(u_t - v_t)$.

Theorem 2 will show that LDU is a social welfare relation and will summarize many of its properties. But we start with a characterization result: motivated by the frequent occurrence of summable or eventually periodic streams[10] in the discussion of utilitarian criteria, we show that three properties of LDU characterize its behavior on the set of streams

$$\mathscr{D} = \{(u, v) \in \mathscr{U} \times \mathscr{U} : u - v \text{ is summable or eventually periodic}\}.$$

The first two properties are traditional assumptions on Pareto improvements and interpersonal comparison of utility. For $u, v \in \mathscr{U}$, write $u > v$ if $u \neq v$ and $u_t \geq v_t$ for all generations t. We define:

Strong Pareto (SP): For all $u, v \in \mathscr{U}$, if $u > v$, then $u \succ v$.

Additivity (Add): For all $u, v, \alpha \in \mathscr{U}$, if $u \succsim v$, then $u + \alpha \succsim v + \alpha$.

Several authors use axioms on interpersonal comparison of utility that are weaker than Additivity.[11] But $u - v = (u + \alpha) - (v + \alpha)$ for all $u, v, \alpha \in \mathscr{U}$, so Additivity holds if weak preference depends only on the difference between streams. This is the case for all SWRs in this paper.

For $u \in \mathscr{U}$ and $c \in \mathbb{R}$, call $(c, u) \equiv (c, u_1, u_2, u_3, \ldots)$ the *(compensated) postponement* of u with *compensation* c.[12] Our third property says that if u has a well-defined average $\bar{u} \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} u_t$, then this average is a reasonable compensation for a postponement of u:[13]

Compensation Principle (CP): For all $u \in \mathscr{U}$ with a well-defined average $\bar{u}$, $u \sim (\bar{u}, u)$.

Why does it use the average as compensation, and not some other number? We motivate this in more detail later in this section: assuming some common axioms, we will argue that — on a fairly large set of streams u — the only compensated postponement (c, u) that is equivalent with u uses the average $c = \bar{u}$ as compensation level. But first, let us observe that these three properties of LDU characterize the order of streams in the domain $\mathscr{D}$. All proofs are in the appendix.

THEOREM 1. *Every social welfare relation that satisfies Strong Pareto, Additivity, and the Compensation Principle coincides with* $\succsim_{\text{LDU}}$ *on* $\mathscr{D}$.

[9] Recall: given $f : (0,1) \to \mathbb{R}$, $\liminf_{x\to 1^-} f(x) = \lim_{\varepsilon \to 0^+} \inf\{f(x) : x \in (1-\varepsilon, 1)\} = \sup_{0<\varepsilon<1} \inf\{f(x) : x \in (1-\varepsilon, 1)\}$.

[10] Stream $u \in \mathscr{U}$ is *summable* if the series $\sum_{t=1}^{\infty} u_t$ converges, and *eventually periodic* (with period p) if there are $k, p \in \mathbb{N}$ with $u_{t+p} = u_t$ for all integers $t \geq k$; if we can take $k = 1$, u is *periodic*. We show later that each pair $(u, v) \in \mathscr{D}$ can be can be compared using LDU; that is, either $u \succsim_{\text{LDU}} v$ or $v \succsim_{\text{LDU}} u$.

[11] These include the Partial Unit Comparability axiom in Basu and Mitra (2007) and Partial and Finite Translation Scale Invariance axiom in Asheim (2010) and Asheim et al. (2010), respectively.

[12] Koopmans et al. (1964) refer to (c, u) as the postponement of u with "insertion" c.

[13] The Compensation Principle can be interpreted in terms of the loss to society due to a compensated postponement. This can be made precise using Abel summation. To elaborate, if $u \in \mathscr{U}$ has a well-defined average and $v = (c, u)$ for some compensation c, the partial sum s_n of the first $n \in \mathbb{N}$ terms of $u - v$ equals $u_n - c$. So the partial sums have average $\bar{s} = \bar{u} - c$. By the Abelian theorem of Frobenius (1880), also the Abel sum $\lim_{\delta\to 1^-} \sigma_\delta(u - v)$ equals $\bar{u} - c$. So in terms of Abel summation, postponing u with compensation $c = 0$ incurs a loss of $\bar{u}$. For example, postponing $u = (1, 1, 1, \ldots)$ incurs a loss of $\bar{u} = 1$ and postponing $u' = (1, 0, 1, 0, \ldots)$ implies a loss of $\bar{u}' = 1/2$. We thank Faruk Gul for this observation.

We sketch the proof of Theorem 1. LDU is a SWR satisfying the three axioms (see Theorem 2), so it suffices to show that they uniquely determine the order of any pair of streams $(u,v)\in\mathscr{D}$. A key step is that for all $u,v\in\mathscr{U}$, the sequence $s=(s_1,s_2,\dots)$ of partial sums $s_n=\sum_{t=1}^{n}(u_t-v_t)$ of $u-v$ satisfies $u-v=s-(0,s)$. So if s is bounded and has a well-defined average $\bar{s}$, the Compensation Principle says that $s\sim(\bar{s},s)$. Additivity and Strong Pareto then give

$$u\succsim v \iff s\succsim(0,s) \iff (\bar{s},s)\succsim(0,s) \iff \bar{s}\geq 0. \tag{2}$$

This addresses the easier cases of the theorem. For instance, if (u,v) in $\mathscr{D}$ has a summable difference $u-v$, its partial sums are bounded and have average $\bar{s}=\sum_{t=1}^{\infty}(u_t-v_t)$. Substitution in (2) leads to an intuitive utilitarian property:

Total Utility property (TU): For all $u,v\in\mathscr{U}$, if $u-v$ is summable, then $u\succsim v$ if and only if $\sum_{t=1}^{\infty}(u_t-v_t)\geq 0$.

But what if $u-v$ is eventually periodic? If its average is zero, then also its sequence s of partial sums is eventually periodic: it has a well-defined average $\bar{s}$ and (2) shows that its sign determines how u and v are ordered. If the average of $u-v$ is not zero, a monotonicity argument can be used to translate it to the zero-average case. We illustrate with a simple example.

Suppose $u-v$ is a periodic stream $(a,b,a,b,\dots)$ for distinct numbers a and b. If its average $(a+b)/2$ is zero, then the partial sum $s_n=\sum_{t=1}^{n}(u_t-v_t)$ equals a for odd n and 0 for even n: the partial sums have average $\bar{s}=a/2$. By (2), u and v are comparable and $u\succsim v$ if and only if $\bar{s}\geq 0$.

And if its average $(a+b)/2$ is not zero, a monotonicity argument makes it possible to use the conclusions from the zero-average case. For instance, if the average is positive, then at least one of a and b, say a, is positive. Define $u'=u-(0,a+b,0,a+b,\dots)$ to obtain $u'-v=(a,-a,a,-a,\dots)$. This periodic stream has average 0 and its partial sums have average $a/2>0$, so by the previous case u' and v are comparable: $u'\succ v$, since $a>0$. Since $u>u'$, Strong Pareto gives $u\succ u'\succ v$, i.e., $u\succ v$: again, our axioms uniquely determine the order between u and v.

We now return to the question why it makes sense to use the average of a stream in our Compensation Principle. To prepare for a more elaborate overview of the properties of LDU in Theorem 2 and to gain a better understanding of our Compensation Principle, we discuss further axioms to explain that for well-behaved streams u, equivalence $(c,u)\sim u$ can only hold if c is the average $\bar{u}$ of u. Anonymity is an equity assumption saying that preferences are unaffected by switching the utility level of any two generations:

Anonymity (Ano): For all $u,v\in\mathscr{U}$, if there are generations $s,t\in\mathbb{N}$ with $u_s=v_t, u_t=v_s$, and $u_i=v_i$ for all other generations $i\in\mathbb{N}$, then $u\sim v$.

Stationarity is Koopmans's (1960) familiar condition for dynamic consistency: preferences are independent of the first generation if this generation receives the same utility in the social states defined by u and v:[14]

Stationarity (Stat): For all $u,v\in\mathscr{U}$ and $c\in\mathbb{R}$, $u\succsim v$ if and only if $(c,u)\succsim(c,v)$.

If a social welfare relation satisfies Strong Pareto, Additivity, Anonymity, and Stationarity, and a stream $u\in\mathscr{U}$ is eventually periodic, then $(c,u)\sim u$ implies that $c=\bar{u}$. Look, for instance, at the periodic stream $u=(a,b,a,b,\dots)$ with $a,b\in\mathbb{R}$. If $(c,u)\sim u$, then Stationarity gives

$$(c,c,a,b,a,b,\dots)\sim(c,a,b,a,b,\dots)\sim(a,b,a,b,\dots).$$

[14]Also Asheim et al. (2010) stress the relevance of Stationarity for intergenerational utilitarianism. They motivate their utilitarian extension of Basu and Mitra's relation (4) by the desirability of retaining Stationarity.

Subtracting $u=(a,b,a,b,\dots)$ from the first and third term and using Additivity gives

$$(c-a,c-b,0,0,\dots)\sim(0,0,0,0,\dots). \tag{3}$$

Now, Strong Pareto, Additivity, and Anonymity imply[15] that for all $u,v\in\mathscr{U}$ differing in at most finitely many coordinates: $u\succsim v$ if and only if their difference $u-v$ has a nonnegative sum. So (3) is equivalent with $(c-a)+(c-b)=0$, i.e., $c=(a+b)/2$, the average of $u=(a,b,a,b,\dots)$.

The insight that $(c,u)\sim u$ implies that $c=\bar{u}$ extends to many other streams u, including all convergent and consequently all summable streams, under a mild continuity assumption. We defer technical details to Proposition 1 in Appendix A.1. This continuity assumption captures the idea from Brock's (1970, p. 929) characterization of the overtaking criterion that "decisions on infinite programs are consistent with decisions on finite programs of length n if n is large enough";[16] for $u\in\mathscr{U}$ and $n\in\mathbb{N}$, write $u_{[n]}=(u_1,u_2,\dots,u_n,0,0,\dots)$:

> *Continuity* (Cont): For all $u,v\in\mathscr{U}$, if there is an $N\in\mathbb{N}$ with $u_{[n]}\succ v_{[n]}$ for all $n\geq N$, then $u\succsim v$.

Up to now, we characterized LDU on the domain $\mathscr{D}$ using Strong Pareto, Additivity, and the Compensation Principle. Moreover, we introduced other axioms to support why this principle uses the average as a compensation for a postponement. So this seems the right time to stress that LDU is a social welfare relation satisfying all these properties:

THEOREM 2. *Limit-discounted utilitarianism defines a social welfare relation. It satisfies Strong Pareto, Additivity, the Compensation Principle, the Total Utility property, Anonymity, Stationarity, and Continuity.*

With the properties in this theorem, it is easier to compare limit-discounted utilitarianism to other utilitarian criteria. That is the topic of our next section.

3. Comparison to other utilitarian criteria

In this section, we compare limit-discounted utilitarianism to three other utilitarian social welfare relations: the criterion $\succsim_{\mathrm{BM}}$ of Basu and Mitra (2007) where

$$u\succsim_{\mathrm{BM}} v \iff \text{there is a } T_0\in\mathbb{N} \text{ with } \sum_{t=1}^{T_0}(u_t-v_t)\geq 0 \text{ and } u_t-v_t\geq 0 \text{ for all } t\geq T_0, \tag{4}$$

the *overtaking* criterion $\succsim_{\mathrm{W}}$ of von Weizsäcker (1965) where

$$u\succsim_{\mathrm{W}} v \iff \text{there is a } T_0\in\mathbb{N} \text{ with } \sum_{t=1}^{T}(u_t-v_t)\geq 0 \text{ for all } T\geq T_0, \tag{5}$$

[15] See Basu and Mitra (2007, Lemma 1) or Jonsson and Voorneveld (2015, Lemma 1).

[16] Continuity is called "horizon consistency" in Jonsson and Voorneveld (2015, p. 23). It is less demanding than Brock's third axiom and the "weak consistency" axiom in Basu and Mitra (2007, p. 359): it allows weak preference rather than demanding that $u\succ v$ hold if $u_{[n]}\succ v_{[n]}$ for large n. Our convention of setting the welfare of generations $t>n$ in $u_{[n]}$ equal to zero follows their papers, but by Additivity, any other constant would do: what matters is that generations $t>n$ receive equal welfare in $u_{[n]}$ and $v_{[n]}$. For instance, if Additivity is satisfied, then Continuity is equivalent with a relaxation of "weak preference continuity" in Asheim and Tungodden (2004, p. 223): for all $u,v\in\mathscr{U}$, if there is an $N\in\mathbb{N}$ with $(u_1,\dots,u_n,v_{n+1},v_{n+2},\dots)\succ v$ for all $n\geq N$, then $u\succsim v$.

and the *catching-up* criterion $\succsim_{\mathrm{G}}$ of Gale (1967) where

$$u \succsim_{\mathrm{G}} v \iff \liminf_{T\to\infty} \sum_{t=1}^{T} (u_t - v_t) \geq 0, \tag{6}$$

for all $u, v \in \mathscr{U}$.[17] Relations and differences between the criteria are discussed along two dimensions: what properties of LDU from the previous section do the other criteria satisfy? And to what extent do preferences according to LDU agree with those of the other criteria?

We summarize some relations before going into details. Properties of LDU were established in Theorem 2 and those of the other criteria are well-known from earlier literature or straightforward to verify; we collect them in Table 1 and will illustrate violations of axioms by means of examples. In Theorem 3 we show that if a stream u is weakly preferred to v according to any of the other three criteria, then the same is true for limit-discounted utilitarianism. For the Basu-Mitra criterion, the implication holds for strict preference as well.

		SP	Add	CP	Ano	Stat	Cont	TU
LDU	$\succsim_{\mathrm{LDU}}$	+	+	+	+	+	+	+
Basu-Mitra	$\succsim_{\mathrm{BM}}$	+	+	−	+	+	−	−
overtaking	$\succsim_{\mathrm{W}}$	+	+	−	+	+	+	−
catching-up	$\succsim_{\mathrm{G}}$	+	+	−	+	+	+	+

Table 1: Social welfare relations and properties they do (+) or do not (−) satisfy.

We now consider a few concrete cases, starting with two examples where the overtaking criterion $\succsim_{\mathrm{W}}$ and the catching-up criterion $\succsim_{\mathrm{G}}$ have been criticized.[18]

EXAMPLE 1. Let $u \in \mathscr{U}$ be a summable stream with strictly positive entries and let $v = (0, u)$. Then $\sum_{t=1}^{T}(u_t - v_t) = u_T > 0$ for all $T \geq 1$. So u is strictly preferred to $v = (0, u)$ under the overtaking criterion. Since the two streams have the same sum, overtaking does not satisfy the Total Utility property. Nor does the Basu-Mitra criterion, which cannot compare u and v.[19] LDU and catching-up do satisfy the Total Utility property: they find u and $(0, u)$ equivalent.

The following example shows that the Compensation Principle and Strong Pareto allow us to compare streams that have appeared frequently in the literature, but which are not comparable using overtaking, catching-up, or the Basu-Mitra criterion.

EXAMPLE 2. Consider the periodic stream $u = (1, 0, 1, 0, \ldots)$ and let $v = (c, u)$ for some real number c. Then

$$\sum_{t=1}^{T}(u_t - v_t) = -\sum_{t=1}^{T}(v_t - u_t) = \begin{cases} -c+1 & \text{if } T \text{ is odd,} \\ -c & \text{if } T \text{ is even.} \end{cases}$$

So u and v are not comparable with the overtaking criterion or the catching-up criterion if $c \in (0, 1)$. Both criteria rank u above v if $c \leq 0$. In particular, they prefer $(1, 0, 1, 0, \ldots)$ to $(0, 1, 0, 1, \ldots)$.

[17] Our definitions of the overtaking and catching-up criterion follow Gale (1967).

[18] Basu and Mitra (2007, p. 361) consider a version of Example 1. Versions of Example 2 have been discussed by, among others, Lauwers (1995, p. 348), Lauwers (1997, p. 225), Fleurbaey and Michel (2003, p. 783), Asheim and Tungodden (2004, p. 229), Heal (2005, p. 1115), Banerjee (2006, p. 333), Basu and Mitra (2007, p. 360), Dutta (2008, Sec. IV), Asheim et al. (2010, p. 520), and Asheim and Banerjee (2010, p. 164).

[19] Since $\sum_{t=1}^{\infty} u_t = \sum_{t=1}^{\infty} v_t$ and $\sum_{t=1}^{T}(u_t - v_t) > 0$ for every T, there is no T_0 with $u_t \geq v_t$ for all $t \geq T_0$. Therefore, $u \succsim_{\mathrm{BM}} v$ does not hold. Likewise, $v \succsim_{\mathrm{BM}} u$ does not hold. This also shows that $\succsim_{\mathrm{BM}}$ violates Continuity: $u_{[n]} \succ_{\mathrm{BM}} v_{[n]}$ for all n, but u and v are not comparable.

Basu and Mitra's criterion does not compare $u = (1,0,1,0,\dots)$ and $v = (c, u)$ for any c, since $u - v = (1-c, -1, 1, -1, \dots)$. So none of these three criteria satisfies the Compensation Principle, which requires that $u \sim (c, u)$ for $c = 1/2$. LDU compares u and v for all c. Indeed, for $\delta \in (0,1)$,

$$\sum_{t=1}^{\infty} \delta^{t-1}(u_t - v_t) = \sum_{t=1}^{\infty}(-\delta)^{t-1} - c = \frac{1}{1+\delta} - c.$$

Letting δ go to one, the Abel sum of $\sum_{t=1}^{\infty}(u_t - v_t)$ is $1/2 - c$. This means that $u \sim_{\text{LDU}} v$ if $c = 1/2$, $u \succ_{\text{LDU}} v$ if $c < 1/2$, and $v \succ_{\text{LDU}} u$ if $c > 1/2$. This can also be seen from the axioms: the Compensation Principle says that $u \sim_{\text{LDU}} v$ if $c = 1/2$; the strict preferences for other c follow from the Strong Pareto axiom.

The next example contains a pair of streams over which catching-up has a strict preference, but LDU is indifferent; Example 1 gave a corresponding result for overtaking.

EXAMPLE 3. Define $u \in \mathscr{U}$ by setting $u_t = 1$ if $t = 2^n$ for some $n \in \mathbb{N}$ and $u_t = 0$ otherwise. Then $u - (0, u) = (u_1, u_2 - u_1, u_3 - u_2, \dots)$ has partial sums $s_n = u_n$, which means that $u \succ_{\text{G}} (0, u)$. But since $\bar{u} = 0$, the Compensation Principle implies that $u \sim_{\text{LDU}} (0, u)$.

Our next theorem states the relations between LDU, overtaking, catching up, and the Basu-Mitra criterion. Given two SWRs $\succsim_A$ and $\succsim_B$ on $\mathscr{U}$, say that $\succsim_B$ *weakly extends* $\succsim_A$ if for all $u, v \in \mathscr{U}$, $u \succsim_A v$ implies $u \succsim_B v$. If, in addition, for all $u, v \in \mathscr{U}$, $u \succ_A v$ implies $u \succ_B v$, we say that $\succsim_B$ *extends* $\succsim_A$.

THEOREM 3. *The following relations hold between the LDU criterion $\succsim_{\text{LDU}}$, the Basu-Mitra criterion $\succsim_{\text{BM}}$, overtaking $\succsim_{\text{W}}$, and catching-up $\succsim_{\text{G}}$:*

(i) $\succsim_{\text{LDU}}$ *extends* $\succsim_{\text{BM}}$.

(ii) $\succsim_{\text{LDU}}$ *weakly extends* $\succsim_{\text{G}}$ *and* $\succsim_{\text{G}}$ *weakly extends* $\succsim_{\text{W}}$.

(iii) $\succsim_{\text{LDU}}$ *does not extend* $\succsim_{\text{W}}$*: there are $u, v \in \mathscr{U}$ with $u \succ_{\text{W}} v$ and $u \sim_{\text{LDU}} v$.*

(iv) $\succsim_{\text{LDU}}$ *does not extend* $\succsim_{\text{G}}$*: there are $u, v \in \mathscr{U}$ with $u \succ_{\text{G}} v$ and $u \sim_{\text{LDU}} v$.*

Throughout the paper we also indicated well-behaved pairs of streams that can be compared using limit-discounted utilitarianism, but not under any of these other criteria: the proof of Theorem 1 and Example 2 use the Compensation Principle to illustrate that all pairs of streams in $\mathscr{D}$ — those whose difference is summable or eventually periodic — are comparable using our criterion. A fortiori, many of our results about streams that can be compared under LDU use a generalization of the theorem of Frobenius (1880), connecting the limit behavior of the discounted sum and the averages of the partial sums. In the terminology of summability criteria, it connects Abel- and Cesàro-summability. For a real sequence $a = (a_1, a_2, \dots)$ and $n \in \mathbb{N}$, let $s_n = \sum_{t=1}^{n} a_t$ be the partial sum of its first n terms and denote their average by

$$C_n(a) = \frac{s_1 + s_2 + \cdots + s_n}{n}. \tag{7}$$

If these averages converge, a is *Cesàro-summable* to $\bar{s} = \lim_{n\to\infty} C_n(a)$. More generally:[20]

[20] The inequalities in Lemma 1 are well-known in the literature on stochastic games; see Lippman (1969), Sennott (1999), and Bishop et al. (2014). These references do not contain a proof of the result in the generality that we stated it. Our proof in the appendix is a slightly rewritten version of one suggested by an anonymous referee.

The paper of Bishop et al. (2014) contains interesting examples illustrating when these inequalities may be strict and shows that such scenarios can occur in practical applications like Markov Decision Processes.

LEMMA 1. *For each* $a = (a_1, a_2, \dots) \in \mathscr{U}$,

$$\liminf_{n\to\infty} C_n(a) \le \liminf_{\delta\to 1^-} \sigma_\delta(a) \le \limsup_{\delta\to 1^-} \sigma_\delta(a) \le \limsup_{n\to\infty} C_n(a). \tag{8}$$

In particular, for $u, v \in \mathscr{U}$, if the partial sums $s_n = \sum_{t=1}^{n}(u_t - v_t)$ of $u - v$ satisfy

$$\liminf_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} \ge 0,$$

then $u \succsim_{\text{LDU}} v$. We conclude with an example of streams that limit-discounted utilitarianism cannot compare.

EXAMPLE 4 (Bishop et al., 2014, Example 2). Define $a = (a_1, a_2, \dots) \in \mathscr{U}$ by

$$a_t = \begin{cases} 0 & \text{if } k! \le t < 2k! \text{ for some } k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

We have (Bishop et al., 2014, Proposition 2):

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} a_t = 1/2 < \liminf_{\delta\to 1^-} (1-\delta)\sigma_\delta(a) = 3/4 \tag{9}$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} a_t = \limsup_{\delta\to 1^-} (1-\delta)\sigma_\delta(a) = 1. \tag{10}$$

If we take $u = a - (0, a)$, then the partial sums of u equal $s_n = \sum_{t=1}^{n} u_t = a_n$, $n \ge 1$. Moreover, summation by parts gives that $\sigma_\delta(u) = (1-\delta)\sigma_\delta(a)$. By (9) and (10),

$$\liminf_{n\to\infty} C_n(u) = 1/2 < \liminf_{\delta\to 1^-} \sigma_\delta(u) = 3/4 \qquad \text{and} \qquad \limsup_{n\to\infty} C_n(u) = \limsup_{\delta\to 1^-} \sigma_\delta(u) = 1.$$

Consequently, u and $v = (r, 0, 0, \dots)$ are not $\succsim_{\text{LDU}}$-comparable if $r \in (3/4, 1)$.

A. APPENDIX

This appendix contains all proofs. They are in a different order than in the text: Proposition 1 in A.1 gives axiomatic support for using the average in our Compensation Principle. A.2 to A.5 contain proofs of Lemma 1 and Theorems 2, 1, and 3, respectively. In the proofs, we refer to axioms using their abbreviations from Section 2.

A.1 *Motivating the average in the Compensation Principle*

We will argue that if a stream u on average gives each generation a utility of $\bar{u}$, then $c = \bar{u}$ is a reasonable compensation in a compensated postponement (c, u) of u: using some of our earlier axioms, Proposition 1 shows — on a fairly large set of well-behaved streams — that $(c, u) \sim u$ can only hold if $c = \bar{u}$. Informally, we will say that a stream $u \in \mathscr{U}$ has a regular average if (i) its average $\bar{u}$ is well-defined and (ii) the average over sufficiently long, but finite segments of consecutive coordinates remains close to $\bar{u}$. Formally, $u \in \mathscr{U}$ has a *regular average* if $\bar{u}$ is well-defined and for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that the average of u over any segment of $n \geq N$ consecutive coordinates differs from $\bar{u}$ by at most ε:

$$\left| \frac{1}{n} \sum_{t=t_0+1}^{t_0+n} u_t - \bar{u} \right| < \varepsilon \qquad \text{for all } t_0 \in \mathbb{N} \text{ and } n \geq N.$$

For instance, the set of streams with a regular average contains all streams that are eventually periodic, summable, or convergent.[21]

PROPOSITION 1. *Let $\succsim$ be a SWR on $\mathscr{U}$ that satisfies Strong Pareto, Anonymity, Additivity, and Stationarity. If $u \in \mathscr{U}$ is eventually periodic and $c \in \mathbb{R}$, then*

(i) $(c,u) \succsim u$ implies $c \geq \bar{u}$.

(ii) $u \succsim (c,u)$ implies $c \leq \bar{u}$.

(iii) $(c,u) \sim u$ implies $c = \bar{u}$.

If the SWR also satisfies Continuity, these implications hold for all $u \in \mathscr{U}$ with a regular average.

PROOF. Recall from footnote 15 that SP, Ano, and Add imply that for all $u, v \in \mathscr{U}$ where $u - v$ has only finitely many nonzero entries:

$$u \succsim v \iff \sum_{t=1}^{\infty} (u_t - v_t) \geq 0. \tag{11}$$

Assume that SWR $\succsim$ on $\mathscr{U}$ satisfies SP, Ano, Add, and Stat. Let $u \in \mathscr{U}$ be eventually periodic and let $c \in \mathbb{R}$. To prove (i), assume that $(c,u) \succsim u$. Since u is eventually periodic, there are $k, p \in \mathbb{N}$ with $u_{t+p} = u_t$ for $t \geq k$. If $p = 1$, then $u = (u_1, \ldots, u_k, u_k, u_k, \ldots)$ is eventually constant, so $(c,u) - u$ has a finite number of nonzero entries that sum to $c - u_k$. Since SP, Ano, and Add are satisfied, (11) implies that $c \geq u_k = \bar{u}$. If $p \geq 2$, we have $(c,c,u) \succsim (c,u) \succsim u$ by Stat and $(c,c,u) \succsim u$ by transitivity. Iterating (if $p > 2$) gives $(\underline{c}_p, u) \succsim u$, where $\underline{c}_p$ is a short-cut notation for p consecutive coordinates equal to c. Since $u_t = (\underline{c}_p, u)_t$ for all $t > k + p$, the difference $(\underline{c}_p, u) - u$ has finitely many nonzero entries and sum $pc - \sum_{t=k+1}^{k+p} u_t$. By (11), $(\underline{c}_p, u) \succsim u$ implies that that this sum is nonnegative: $c \geq \sum_{t=k+1}^{k+p} u_t / p = \bar{u}$.
(ii): By Add, $u \succsim (c,u)$ implies $(-c, -u) \succsim -u$, which by (i) implies $-c \geq -\bar{u}$, so that $c \leq \bar{u}$.
(iii): This follows from (i) and (ii).

Now add axiom Cont. We prove implication (i) for streams with a regular average; (ii) and (iii) follow as above. So let $u \in \mathscr{U}$ and $c \in \mathbb{R}$ be such that u has regular average $\bar{u}$ and $(c,u) \succsim u$. By Stat and transitivity, $(\underline{c}_N, u) \succsim u$ for all $N \in \mathbb{N}$. To prove that $c \geq \bar{u}$, suppose, to the contrary, that $c < \bar{u}$. We obtain a contradiction by showing that $u \succ (\underline{c}_N, u)$ for some $N \in \mathbb{N}$.

If $c < \bar{u}$, there are $b \in \mathbb{R}$ with $c < b < \bar{u}$ and $N \in \mathbb{N}$ such that the average of u over any $n \geq N$ consecutive generations differs from $\bar{u}$ by at most $\varepsilon \equiv (\bar{u} - b)/2$. Let $d = u - (\underline{b}_N, u)$. For $n > N$:

$$\begin{aligned}
\sum_{t=1}^{n} d_t &= u_n + u_{n-1} + \cdots + u_{n-N+1} - Nb \\
&= N((u_n + u_{n-1} + \cdots + u_{n-N+1})/N - b) \\
&\geq N(\bar{u} - \varepsilon - b) \\
&= N\varepsilon.
\end{aligned}$$

[21] A stream with an average that is not regular is $u = (0,1,0,0,1,1,0,0,0,1,1,1,\ldots)$, consisting of a zero followed by a one, then two zeros followed by two ones, three zeros followed by three ones, etc. Its average is $1/2$. But it does not have a regular average: for every n, the stream contains infinitely many segments of n consecutive zeros (and n consecutive ones).

Since $d_{[n]}$ has finitely many nonzero entries and positive sum, (11) implies that $d_{[n]} \succ (0,0,0,\ldots)$ for all $n > N$. By Cont, we have $d \succsim (0,0,0,\ldots)$. Hence $u \succsim (\underline{b}_N, u)$ by Add. By SP and $b > c$: $u \succ (\underline{c}_N, u)$, which is our contradiction. □

A.2 *Proof of Lemma 1*

We prove Lemma 1 slightly more generally, for all real sequences $a = (a_1, a_2, \ldots)$ whose discounted sum $\sigma_\delta(a)$ is well-defined for each $\delta \in (0,1)$. Let $s_n = \sum_{t=1}^{n} a_t, n \in \mathbb{N}$. In (8), note that all upper/lower limits are well-defined in $\mathbb{R} \cup \{-\infty, +\infty\}$ and that the second inequality holds: all infima and suprema are taken over nonempty sets. The first inequality in (8) implies the third using a sign change: $\limsup_{\delta\to 1^-} \sigma_\delta(a) = -\liminf_{\delta\to 1^-} \sigma_\delta(-a)$, $\limsup_{n\to\infty} C_n(a) = -\liminf_{n\to\infty} C_n(-a)$. So it suffices to prove the first inequality:

$$\liminf_{n\to\infty} C_n(a) \leq \liminf_{\delta\to 1^-} \sigma_\delta(a).$$

Summation by parts, first for the sequence a and then for its partial sums, gives

$$\begin{aligned}\sum_{n=1}^{\infty} \delta^{n-1} a_n &= a_1 + \sum_{n=2}^{\infty} \delta^{n-1}(s_n - s_{n-1}) \\ &= (1-\delta) \sum_{n=1}^{\infty} \delta^{n-1} s_n \\ &= (1-\delta)^2 \sum_{n=1}^{\infty} \delta^{n-1}(s_1 + \cdots + s_n).\end{aligned}$$

So, recalling from (7) that $C_n(a) = (s_1 + \cdots + s_n)/n$, the discounted sum equals

$$\sigma_\delta(a) = \sum_{n=1}^{\infty} \delta^{n-1} a_n = (1-\delta)^2 \sum_{n=1}^{\infty} \delta^{n-1} n C_n(a). \tag{12}$$

Distinguish three cases. Firstly, if $\liminf_{n\to\infty} C_n(a) = +\infty$, then (12) and the equality

$$\sum_{n=1}^{\infty} n\delta^{n-1} = \frac{1}{(1-\delta)^2} \tag{13}$$

immediately give that also $\liminf_{\delta\to 1^-} \sigma_\delta(a) = +\infty$. Secondly, if $\liminf_{n\to\infty} C_n(a) = -\infty$, the inequality $\liminf_{n\to\infty} C_n(a) \leq \liminf_{\delta\to 1^-} \sigma_\delta(a)$ holds trivially, since both lower limits are well-defined. Finally, suppose that $\liminf_{n\to\infty} C_n(a)$ is finite. For $\lambda \in \mathbb{R}$, (12) and (13) give

$$\sum_{n=1}^{\infty} \delta^{n-1} a_n - \lambda = (1-\delta^2) \sum_{n=1}^{\infty} \delta^{n-1} n(C_n(a) - \lambda).$$

Take $\lambda = \liminf_{n\to\infty} C_n(a)$. By definition of λ, for each $\varepsilon > 0$ there is a T such that $C_n(a) - \lambda \geq -\varepsilon$ for all $n > T$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \delta^{n-1} a_n - \lambda &\geq (1-\delta^2) \sum_{n=1}^{T} \delta^{n-1} n(C_n(a) - \lambda) - \varepsilon(1-\delta^2) \sum_{n=T+1}^{\infty} \delta^{n-1} n \\ &\geq (1-\delta^2) \sum_{n=1}^{T} \delta^{n-1} n(C_n(a) - \lambda) - \varepsilon.\end{aligned} \tag{14}$$

The first term in (14) tends to 0 as $\delta \to 1^-$, so $\liminf_{\delta\to 1^-} \sigma_\delta(a) - \lambda \geq -\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\liminf_{\delta\to 1^-} \sigma_\delta(a) \geq \lambda = \liminf_{n\to\infty} C_n(a)$.

A.3 *Proof of Theorem 2*

We first show that $\succsim_{\text{LDU}}$ is a social welfare relation. It is reflexive: $\sigma_\delta(u-u)=\sigma_\delta(0,0,\dots)=0$ for all $u\in\mathcal{U}$ and $\delta\in(0,1)$. It is transitive: let $u,v,w\in\mathcal{U}$ have $u\succsim_{\text{LDU}} v$ and $v\succsim_{\text{LDU}} w$. For each $\delta\in(0,1)$, the discounted sums satisfy

$$\sigma_\delta(u-w)=\sigma_\delta(u-v)+\sigma_\delta(v-w),$$

so taking lower limits and using $u\succsim_{\text{LDU}} v$ and $v\succsim_{\text{LDU}} w$ gives

$$\liminf_{\delta\to1^-}\sigma_\delta(u-w)\geq\liminf_{\delta\to1^-}\sigma_\delta(u-v)+\liminf_{\delta\to1^-}\sigma_\delta(v-w)\geq0+0=0,$$

i.e., $u\succsim_{\text{LDU}} w$. We proceed to the axioms:
SP: If $u>v$, the discounted sum $\sigma_\delta(u-v)$ is a positive, increasing function of $\delta\in(0,1)$. So $\lim_{\delta\to1^-}\sigma_\delta(u-v)$ exists in $(0,+\infty]$ and $\lim_{\delta\to1^-}\sigma_\delta(v-u)=-\lim_{\delta\to1^-}\sigma_\delta(u-v)\in[-\infty,0)$. So $u\succ_{\text{LDU}} v$.
Add: For all $u,v,\alpha\in\mathcal{U}$, $(u+\alpha)-(v+\alpha)=u-v$.
CP: Let $u\in\mathcal{U}$ have a well-defined average $\bar{u}$ and let $c\in\mathbb{R}$. We show that the discounted sum of $d=u-(c,u)=(u_1-c,u_2-u_1,u_3-u_2,\dots)$ satisfies

$$\lim_{\delta\to1^-}\sigma_\delta(u-(c,u))=\bar{u}-c. \tag{15}$$

By Frobenius's theorem (cf. Lemma 1), it suffices to show that the series $\sum_{t=1}^\infty d_t$ is Cesàro-summable to $\bar{u}-c$, i.e., that its partial sums $s_n=\sum_{t=1}^n d_t$ satisfy

$$\frac{s_1+\cdots+s_n}{n}\to\bar{u}-c\text{ as }n\to\infty.$$

But that is easy: the partial sum of the first $n\in\mathbb{N}$ terms is $s_n=u_n-c$, so

$$\frac{s_1+\cdots+s_n}{n}=\frac{u_1+\cdots+u_n}{n}-c.$$

Since $\bar{u}$ exists by assumption, the right-hand side tends to $\bar{u}-c$ as $n\to\infty$. This proves (15). It follows that $(c,u)\sim_{\text{LDU}} u$ if $c=\bar{u}$.
TU: If $u-v$ is summable, then $\sum_{t=1}^\infty(u_t-v_t)=\lim_{\delta\to1^-}\sigma_\delta(u-v)$ by Abel's theorem; cf. Duren (2012, p. 76).
Ano: If $v\in\mathcal{U}$ is obtained from $u\in\mathcal{U}$ by permuting two coordinates, then $\sum_{t=1}^\infty(u_t-v_t)=0$. So Ano follows from TU.
Stat: For all $u,v\in\mathcal{U}$, $\liminf_{\delta\to1^-}\sigma_\delta(u-v)=(u_1-v_1)+\liminf_{\delta\to1^-}\sigma_\delta((u_2,u_3,\dots)-(v_2,v_3,\dots))$.
Cont: Let $u,v\in\mathcal{U}$ and $N\in\mathbb{N}$ be such that $u_{[n]}\succ_{\text{LDU}} v_{[n]}$ for all $n\geq N$. Since $u_{[n]}$ and $v_{[n]}$ are summable, $s_n=\sum_{t=1}^n(u_t-v_t)>0$ for all $n\geq N$ by TU. By Lemma 1, $\liminf_{\delta\to1^-}\sigma_\delta(u-v)\geq0$, i.e., $u\succsim_{\text{LDU}} v$.

A.4 *Proof of Theorem 1*

We start with a lemma linking Strong Pareto, Additivity, and the Compensation Principle to summability:

LEMMA 2. *Let the SWR $\succsim$ satisfy Strong Pareto, Additivity, and the Compensation Principle. For $u,v\in\mathcal{U}$, if the series $\sum_{t=1}^\infty(u_t-v_t)$ is Cesàro-summable and has bounded partial sums, then*

$$u\succsim v\iff\lim_{n\to\infty}C_n(u-v)\geq0. \tag{16}$$

In particular, $\succsim$ has the Total Utility property.

PROOF. For u and v as in the lemma, Add gives $u \succsim v$ if and only if $u - v \succsim (0,0,\dots)$. The sequence $s = (s_1, s_2, \dots)$ of partial sums $s_n = \sum_{t=1}^{n}(u_t - v_t), n \in \mathbb{N}$, is bounded and $u - v = s - (0,s)$. So

$$u \succsim v \iff s - (0,s) \succsim (0,0,\dots). \tag{17}$$

By CP, $s \sim (\bar{s}, s)$, where $\bar{s} = \lim_{n\to\infty} C_n(u-v)$ by definition (7). By Add, $s-(0,s) \sim (\bar{s},s)-(0,s)$. Since

$$(\bar{s},s)-(0,s) = (\bar{s},0,0,\dots),$$

equation (17) implies

$$u \succsim v \iff (\bar{s},0,0,\dots) \succsim (0,0,0,\dots). \tag{18}$$

By reflexivity and SP, $(\bar{s},0,0,\dots) \succsim (0,0,0,\dots)$ holds if and only if $\bar{s} \geq 0$. With (18), this gives (16).

For TU, if $\sum_{t=1}^{\infty}(u_t - v_t)$ is convergent, it is Cesàro-summable to the same sum. That is, s is bounded ($s \in \mathscr{U}$) and $\bar{s} = \sum_{t=1}^{\infty}(u_t - v_t)$. By (16), $u \succsim v$ if and only if $\sum_{t=1}^{\infty}(u_t - v_t) \geq 0$. □

This leaves us properly equipped for the proof of Theorem 1:

PROOF. Let SWR $\succsim$ satisfy SP, Add, and CP. Let $(u,v) \in \mathscr{D}$. If $d \equiv u - v$ is summable, the Total Utility property (Theorem 2 for $\succsim_{\text{LDU}}$, Lemma 2 for $\succsim$) implies that $u \succsim v$ if and only if $u \succsim_{\text{LDU}} v$.

So assume that d is eventually periodic: there are $p, T \in \mathbb{N}$ with $d_{t+p} = d_t$ for all $t \geq T$. For all $t \geq T$, we then have $\bar{d} = \sum_{i=t+1}^{t+p} d_i/p$, i.e., $p\bar{d} = \sum_{i=t+1}^{t+p} d_i$.

Case 1: If $\bar{d} = 0$, then $\sum_{i=t+1}^{t+p} d_i = 0$ for all $t \geq T$. Then $s_n = \sum_{t=1}^{n} d_t, n \geq 1$, is eventually periodic, so that s is bounded and $\bar{s}$ is well-defined. Since both $\succsim_{\text{LDU}}$ and $\succsim$ satisfy the conditions of Lemma 2,

$$u \succsim v \iff \bar{s} \geq 0 \iff u \succsim_{\text{LDU}} v.$$

Case 2: If $\bar{d} \neq 0$, let's suppose that $\bar{d} > 0$: the argument when $\bar{d} < 0$ is similar. If $\bar{d} > 0$, the partial sums $s_n = \sum_{t=1}^{n} d_t$ and hence their averages $C_n(d) = \frac{s_1+\cdots+s_n}{n}$ diverge to $+\infty$. By Lemma 1, $\liminf_{\delta\to1^-} \sum_{t=1}^{\infty} \delta^{t-1} d_t = +\infty$, i.e., $u \succ_{\text{LDU}} v$. It remains to verify that also $u \succ v$.

Since $d = u - v$ is bounded, there is an $M \in [0,\infty)$ such that $-M \leq d_t \leq M$ for all $t \in \mathbb{N}$. Then also $-M \leq \bar{d} \leq M$. Because $\bar{d} > 0$ implies $s_n \to +\infty$, we can choose $N \geq T$ with $s_N \geq 2pM$. We abbreviate $k \in \mathbb{N}$ consecutive zero coordinates by $\underline{0}_k$ and define $u' = u - (\underline{0}_N, p\bar{d}, \underline{0}_{p-1}, p\bar{d}, \underline{0}_{p-1}, p\bar{d}, \underline{0}_{p-1}, \dots)$. Since $u - v$ and $(\underline{0}_N, p\bar{d}, \underline{0}_{p-1}, p\bar{d}, \underline{0}_{p-1}, p\bar{d}, \underline{0}_{p-1}, \dots)$ are eventually periodic with period p and average $\bar{d}$, stream $u' - v$ is eventually periodic with period p and average 0. Arguing as in Case 1, its sequence of partial sums $s'_n = \sum_{t=1}^{n}(u'_t - v_t), n \geq 1$, is bounded, i.e., $s' = (s'_1, s'_2, \dots) \in \mathscr{U}$, and eventually periodic with period p. We claim that $s'_n \geq 0$ for all $n > N$. By periodicity, it suffices to show that $s'_{N+k} \geq 0$ for all $k = 1, \dots, p$. By construction,

$$s'_{N+k} = s_N - p\bar{d} - \sum_{m=1}^{k}(u_{N+m} - v_{N+m}) \geq 2pM - pM - kM \geq 0.$$

Since $s'_n \geq 0$ for all $n > N$, $\bar{s}' \geq 0$. By CP, $s' \sim (\bar{s}', s')$, so $s' \succsim (0, s')$ by SP. By Add, $u' \succsim v$. By SP, $u \succ u'$. By transitivity, $u \succ v$. □

A.5 *Proof of Theorem 3*

(i): Since $\succsim_{\text{LDU}}$ satisfies SP, Ano, and Add, it extends $\succsim_{\text{BM}}$; see Basu and Mitra (2007, Theorem 1) or Jonsson and Voorneveld (2015, Theorem 6).
(ii): By definition, $\succsim_{\text{G}}$ weakly extends $\succsim_{\text{W}}$. To see that $\succsim_{\text{LDU}}$ extends $\succsim_{\text{G}}$, let $u, v \in \mathscr{U}$ have $u \succsim_{\text{G}} v$: the partial sums $s_n = \sum_{t=1}^{n}(u_t - v_t)$ of $u - v$ satisfy $\liminf_{n\to\infty} s_n \geq 0$. Consequently,

$$\liminf_{n\to\infty} C_n(u-v) = \liminf_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} \geq 0.$$

By Lemma 1, also $\liminf_{\delta\to 1^-} \sigma_\delta(u-v) \geq 0$. That is, $u \succsim_{\text{LDU}} v$.
Finally, (iii) was illustrated in Example 1, and (iv) in Example 3.

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Supplement to "The limit of discounted utilitarianism"

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An anonymous referee suggested finding a variant of our characterization (Theorem 1)[1] that does not use the Compensation Principle. This is possible since the Compensation Principle can be split into two parts: (1) a "critical level" assumption: well-behaved streams u have an equivalent compensated postponement (c, u) for some level c and (2) the level c that achieves this equivalence is the average of u. Proposition 1 of our paper shows how to deduce the latter from axioms that do not rely on summation. Together with a critical level assumption, they can replace the Compensation Principle in our characterization:

THEOREM. *If SWR $\succsim$ satisfies Strong Pareto, Additivity, Anonymity, Stationarity, Continuity, and*

$$\text{for all convergent or eventually periodic } u \in \mathscr{U} \text{ there is a } c \in \mathbb{R} \text{ with } u \sim (c, u), \tag{1}$$

then $\succsim$ and $\succsim_{\text{LDU}}$ coincide on $\mathscr{D}$.

PROOF. The proof requires only minor adjustments from that of Theorem 1. Let the SWR $\succsim$ satisfy SP, Add, Ano, Stat, Cont, and (1). Let $(u, v) \in \mathscr{D}$.

If $u - v$ is summable, its sequence $s = (s_1, s_2, \ldots)$ of partial sums converges. This means that s is bounded and its average $\bar{s} = \sum_{t=1}^{\infty}(u_t - v_t)$ is regular. By (1), there is a $c \in \mathbb{R}$ with $s \sim (c, s)$. Proposition 1 gives $c = \bar{s}$, so that $s \sim (\bar{s}, s)$. By Add, $s - (0, s) \sim (\bar{s}, 0, 0, \ldots)$. Thus

$$\begin{aligned}
u \succsim v &\iff u - v \succsim (0,0,\ldots) && \text{(by Add)}\\
&\iff s - (0,s) \succsim (0,0,\ldots) && \text{(since } u - v = s - (0,s))\\
&\iff (\bar{s},0,0,\ldots) \succsim (0,0,\ldots) && \text{(since } s-(0,s) \sim (\bar{s},0,0,\ldots))\\
&\iff \bar{s} \geq 0 && \text{(by SP and reflexivity)}\\
&\iff \textstyle\sum_{t=1}^{\infty}(u_t - v_t) \geq 0 && \text{(since } \bar{s} = \textstyle\sum_{t=1}^{\infty}(u_t - v_t))\\
&\iff u \succsim_{\text{LDU}} v && \text{(since } \succsim_{\text{LDU}} \text{ satisfies TU)}
\end{aligned}$$

If $u - v$ is eventually periodic, the corresponding part of the proof of Theorem 1 carries over almost verbatim. In Case 1, the sequence s of partial sums is eventually periodic, so s is bounded and its average $\bar{s} = \lim_{n \to \infty} C_n(u - v)$ is well-defined. By Lemma 2, $u \succsim_{\text{LDU}} v$ if and only if $\bar{s} \geq 0$. And arguing as above:

$$u \succsim v \iff (\bar{s}, 0, 0, \ldots) \succsim (0, 0, \ldots) \iff \bar{s} \geq 0.$$

So $u \succsim v$ if and only if $u \succsim_{\text{LDU}} v$. Case 2 is unchanged, except the sentence "By CP . . . ": instead of using CP, we apply the reasoning of Case 1 to s' to show that $s' \sim (\bar{s}', s')$. □

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[1]The social welfare relation that ranks $u \succsim v$ whenever $\liminf_{n \to \infty} C_n(u - v) \geq 0$, sometimes called Veinott's criterion, coincides with $\succsim_{\text{LDU}}$ on $\mathscr{D}$ and can consequently be characterized the same way.