Piketty for the Pedestrian

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1.

Introduction

Introduction

In 2013 there was a big commotion concerning the (700 pages!) book

Piketty, T. "Capital in the Twenty-First Century"

This book has been on the best seller lists ever since, and has been the subject of a heated debate.

What is the book and all this fuss about?

Contents of the book

The book is based on 20 years of empirical research concerning the distribution of wealth and income during the last two centuries in the western world.

The book contains

- Empirical results.
- Theoretical results.
- Policy recommendations.

The really important parts of the book are the empirical studies. These have been carried out by a large number of researchers. In Sweden we have Jesper Roine at Handelshögskolan and Daniel Waldenström in Uppsala.

Concerning readability

Despite the fact that the book is 700 pages long, it is extremely well written and easy to read. It is highly entertaining with lots of discussions referring to history and to the classical novels of the 19:th century, such as Austen, Balzac, Tolstoy etc.

Empirical work

The empirical work, which is the main part of the book, focuses on the following two items:

- The income share of the top decile and top percentile.
- Capital's share of total income.

The empirical conclusions are roughly the following:

- The income share of the top decile was decreasing between 1900 and 1980. It has been increasing since 1980.
- Capital's share of total income was decreasing between 1900 and 1980. It has been increasing since 1980.

Theoretical work

Piketty argues, in a rather informal and intuitive way, that the following hold.

- In the long run we should expect a low growth rate.
- Historically, the return of capital has been larger than the growth rate.
- Therefore we should expect that capital's share of income is **increasing** in the long run.
- Economic inequality between capital and labor is thus likely to increase.
- The decrease in capital's share of income from 1900 to 1980 is attributed to the wars.

For the more theoretical parts, Piketty refers to "**the first and second fundamental laws of capitalism**". More about these later.

Policy recommendations

If you feel that substantial economic inequality between income brackets and between capital and labor is a bad thing, then Piketty has the following recommendations.

- Confiscatory income taxes for high income brackets.
- Confiscatory taxes on inherited capital.

Piketty makes it quite clear that, in his opinion, it is not realistic to believe that these recommendations can be carried out in practice.

The Debate on Piketty

The appreciation of Piketty's book often seems to depend on the political view of the readers.

- The political left wing has hailed the Piketty and his book as the second coming of Karl Marx and Das Kapital. "Piketty is an obvious candidate for the Nobel Price".

- The political right wing has claimed that the empirical work is seriously flawed, that the analysis is bad, and they have disliked the policy recommendations with a vengeance.

Piketty's general opinions

- Piketty seems to be happy with they way in which capitalism allocates **resources**. Hard core socialism is **not** the solution for P.

- He is, however, very unhappy about the way in which capitalism allocates **wealth**.

- He is not impressed by current academic practice in economics. Too much focus on narrow journal articles. Too much mathematics. Not enough focus on the big picture. There should be more interplay with psychology, sociology, history etc.

Academic debate

On the academic side there has been two objects of debate.

- Are the Piketty empirical results correct? Shortly after the publication of the book, Financial Times published a very aggressive article claiming that the empirical investigation was seriously incoorect.

- Are the theoretical results correct? In particular there has been a debate concerning Piketty's "second law of capitalism".

Academic consensus

The consensus in academia now seems to be as follows.

- The empirical work is essentially correct. The book is in fact a very important contribution to our understanding of the historical development of income and capital distribution.
- The theoretical arguments, in particular the "second law of capitalism" is heavily criticized.

A one-liner conclusion

The following should be enough to survive a discussion at the pub:

> "Piketty? Well, it is of course a very impressive piece of empirical research, but I am rather sceptical about the second fundamental law."

The object of this talk

In this talk I will focus on the theoretical aspects of the Piketty book. In particular we will study the following points.

- A Kindergarten version of the Piketty theory, including the first and second fundamental laws.
- A more advanced version, based on the classical Solow growth model.
- A "Piketty version" of the Solow model.
- Comparing the classical Solow model with the Piketty version.

2.

The Simplest Growth Model

Basic notation

We use the the following notation with the dimensions in brackets. Time is denoted by $[t]$, and money by $[m]$. The dimension of a dimensionless entity is denoted by $[1]$.

$$
\begin{aligned}
K &= \text{capital } [m] \\
Y &= \text{income per unit time } [m/t] \\
Y^K &= \text{capital income per unit time } [m/t] \\
r &= \text{return rate on capital } [1/t] \\
\alpha &= \text{capital income as a share of total income } [1] \\
\beta &= \text{capital-income ratio } [t]
\end{aligned}
$$

By definition these objects are related as follows

$$
Y^K = rK, \quad \alpha = \frac{Y^K}{Y}, \quad \beta = \frac{K}{Y}
$$

The First Fundamental Law of Capitalism

We can now derive Piketty's "first fundamental law".

Proposition: (The First Fundamental Law)

$$\alpha = r\beta.$$

Proof: This follows directly from

$$\alpha = \frac{Y^K}{Y}, \quad Y^K = rK, \quad \beta = \frac{K}{Y} \quad \blacksquare$$

In words: Capital's share of income **equals** return on capital **times** the capital-income ratio.

The first law is essentially a definition and thus not a big deal.

A word of warning

Since capital income Y^K is included in gross income Y, we must of course have $Y^K \leq Y$, implying that

$$\alpha = \frac{Y^K}{Y} \leq 1$$

This implies in particular that the parameters above cannot be chosen freely. The relevant restrictions on parameters can only be determined within a full fledged equilibrium model.

A Kindergarten model & the Second Law

We define the dynamics of the capital stock K and income Y as

$$\begin{aligned} \dot{K}_t &= sY_t, \\ \dot{Y}_t &= gY_t. \\ s &= \text{savings rate } [1] \\ g &= \text{growth rate of income } [1/t] \end{aligned}$$

We are now ready to derive the second law. From $\beta = K/Y$ we have

$$\dot{\beta}_t = \frac{Y_t \dot{K}_t - K_t \dot{Y}_t}{Y_t^2}$$

Using the dynamics above we easily obtain

$$\dot{\beta}_t = s - g\beta_t.$$

This is a stable ODE, and we immediately have the following result.

The Second Law

Proposition:(Second Fundamental Law of Capitalism) Assume that the savings rate s and the growth rate g are constant over time. Then the capital-income ratio $\beta_t = K_t/Y_t$ converges to an equilibrium value given by

$$\beta = \frac{s}{g}.$$

Equilibrium Capital Share of Income

From the First Law, i.e. the definition of the capital share of income as

$$\alpha = r\beta$$

and the Second Law

$$\beta = \frac{s}{g}$$

we have the following result.

Proposition:(Equilibrium Capital Share of Income) In steady state, the capital share of income, α, is given by

$$\alpha = \frac{sr}{g}.$$

Economic and political conclusions

Piketty's argument is now roughly as follows.

- From econometric data it seems that the return on capital is greater than the growth rate.

- This implies that capital accumulates at a faster rate than the growth rate of income, so the income share due to capital will increase.

- We see indeed from

$$\alpha = s \cdot \left(\frac{r}{g} \right).$$

  that, by doing comparative statics, a high ratio r/g will imply a high capital share of income α.

- In particular we see that if the growth rate approaches zero, then the capital share of income may explode.

- The politically hot implication of this is that we will have increasing income inequality between capital and labor.

- It is in fact obvious how to interpret the Piketty conclusion in Marxist class terms. According to that interpretation, and with the extreme value $\alpha = 1$, the capitalist class will in steady state get more or less all income, while the working class will get nothing above some minimum subsistence level.

Example

- Set $g = 3\%$, $s = 20\%$, and $r = 5\%$ which are reasonable values for Western Europe.

- The steady state capital-income ratio is then given by $\beta = 6.6$ and the capital share of income is $\alpha = 33\%$.

- If we now assume that the growth rate decreases to 0.5% we have a change of the capital-income ratio from 6.6 to 40 and the capital share of income is increased from 33% to 80%. This is indeed quite dramatic.

Counter arguments

The results above have all been widely and hotly debated. The most common counter arguments center around one or several of the following topics.

- The distribution of capital in Western society.
- Capital depreciation.
- The assumptions of constant savings rate, growth rate, and return on capital.

Distribution of capital

One counterargument to a Marxist interpretation goes as follows.

- In modern society capital is owned, not only by traditional “capitalists”, but also to a large extent by the state and by “ordinary people”. A wage earner in western society will have some capital invested in funds (pension funds and others) and he/she may also have capital invested in real estate.

- Since capital income thus will be distributed among the entire population, it is not at all clear that we can equate a large α with large income inequality.

Capital depreciation

Another very obvious counter argument is that Piketty has not taken **capital depreciation** into account. We thus define

$$\delta = \text{capital depreciation rate } [1/t],$$

which implies that the capital growth dynamics are now slightly changed so we ahve

$$\begin{aligned} \dot{K}_t &= sY_t - \delta K_t \\ \dot{Y}_t &= gY_t. \end{aligned}$$

A simple calculation gives us

$$\dot{\beta}_t = s - (g + \delta)\,\beta_t,$$

and we have the following result.

Capital depreciation ct'd

Proposition: With a capital depreciation rate δ the steady state value of the capital income ratio β is given by

$$\beta = \frac{s}{g + \delta}$$

and the equilibrium capital share of income is given by

$$\alpha = s\frac{r}{g + \delta}.$$

We see that the introduction of the depreciation rate δ has a profound effect on the Piketty result: Even if the growth rate g goes to zero, the capital share of income, α, will **not** explode.

Example

- Set $g = 2\%$, $s = 30\%$, $\delta = 5\%$ and $r = 5\%$.
- This gives us a capital-income ratio of $\beta = 4.2$ and the capital share of income is $\alpha = 21\%$.
- If we now decrease g drastically to 0% we obtain $\beta = 6$ and $\alpha = 30\%$
- The increase in β and α are not very dramatic, compared to the case studied above where we had no capital depreciation.

Critique of the simple model

We have assumed that all parameters are constant. This is, however, not very realistic.

- From a microeconomic point of view it is reasonable to expect that the return on capital r decreases as the capital stock K increases.
- The savings rate s should likewise not be independent of g. In particular one may argue that if g goes to zero so we have no growth, then the savings rate also goes to zero, so again we cannot do naive comparative statics.

These issues can only be resolved within a more complete model.

3.

The Solow growth model

The Solow growth model

Model assumptions:
We introduce the following new notation, where $[p]$ denotes "person".

$$\begin{aligned} I_t &= \text{investment rate } [m/t] \\ L_t &= \text{size of labor force } [p]. \end{aligned}$$

The main player in the model is the **production function**

$$F : R^2 \rightarrow R$$

The interpretation is that for a given capital stock K and labor force L, we produce an income rate given by

$$Y = F(K, L)$$

We also need some assumptions concerning the production function, savings, and the evolution of the labor force.

Assumptions

- The production function exhibits constant returns to scale, i.e.

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

  We also assume that F is increasing and concave in K with $F(0, L) = 0$ for all K, and we assume that F satisfies the usual Inada conditions.

$$\frac{\partial F}{\partial K}(0, L) = +\infty, \qquad \lim_{K \to \infty} \frac{\partial F}{\partial K}(K, L) = 0.$$

- Investment (in new capital) is a constant fraction of output, so that

$$I = sF(K, L)$$

  where s is the (constant) savings rate.

- The capital depreciation rate δ is constant over time.

- Labor grows exponentially at the growth rate g, so

$$L_t = e^{gt} L_0.$$

Dynamics

From the assumptions, the dynamics of K, L and Y are given by

$$\begin{aligned} \dot{K}_t &= sF(K_t, L_t) - \delta K_t, \\ Y_t &= F(K_t, L_t), \\ \dot{L}_t &= gL_t. \end{aligned}$$

In order to study this system we introduce the capital-labor ratio k and the income-labor ratio y by

$$k_t = \frac{K_t}{L_t}, \quad y_t = \frac{Y_t}{L_t}.$$

We then obtain

$$\begin{aligned} \dot{k}_t &= sf(k_t) - (\delta + g)\, k_t, \\ y_t &= f(k_t), \end{aligned}$$

where f is defined by

$$f(x) = F(x, 1).$$

Equilibrium

The Inada conditions implies that the fundamental equation

$$\dot{k}_t = sf(k_t) - (\delta + g)\, k_t,$$

is stable.

- The capital-labor ratio k_t will converge to the equilibrium value k^* which the unique solution of the equilibrium equation

$$sf(k^*) = (\delta + g)\, k^*. \tag{1}$$

- The income-labor ratio will converge to the equilibrium value

$$y^* = f(k^*). \tag{2}$$

- The capital-income ratio $\beta = K/Y = k/y$ will converge to the equilibrium value

$$\beta^* = \frac{s}{g + \delta} \tag{3}$$

Labor and capital incomes

We now go on to study the labor wage, i.e. the price of labor, w^L, and the price of capital, w^K. According to standard microeconomic theory we should have

$$\begin{aligned} w_t^K &= F_K(K_t, L_t), \\ w_t^L &= F_L(K_t, L_t). \end{aligned}$$

The labor income per unit time, Y^L, and the capital income per unit time, Y^K, are thus given by

$$\begin{aligned} Y_t^L &= L_t F_L(K_t, L_t), \\ Y_t^K &= K_t F_K(K_t, L_t), \end{aligned}$$

and we note (with some satisfaction) that by the Euler formula $F = KF_K + LF_L$ for linearly homogeneous functions we have

$$Y_t = Y_t^L + Y_t^K.$$

The return on capital, r, is defined by the relation

$$Y_t^K = r_t \cdot K_t$$

so from $Y_t^K = K_t F_K(K_t, L_t)$ we have

$$r_t = F_K(K_t, L_t).$$

Since F is homogeneous we also have

$$F_K(K, L) = F_K(k, 1)$$

with $k = K/L$ as above, so we obtain

Proposition The return on capital is given by

$$r_t = f'(k_t),$$

and in equilibrium we have

$$r^* = f'(k^*)$$

where k^* is given by

$$sf(k^*) = (\delta + g)\, k^*$$

Determining capital's share of income

To compute capital's share of income, α, we have by definition

$$\alpha_t = \frac{Y_t^K}{Y_t}$$

so we obtain

$$\alpha_t = \frac{K_t}{Y_t} F_K(K_t, L_t) = \frac{K_t}{Y_t} f'(k_t) = \frac{K_t}{Y_t} r_t.$$

We now have the following result.

Proposition In steady state the capital share of income is constant and given by

$$\alpha^* = \frac{r^* s}{g + \delta},$$

where the steady state return of capital is given by

$$r^* = f'(k^*).$$

A closer look at α

We work in steady state and recall that

$$\alpha = \frac{rs}{g + \delta} \qquad (4)$$

where the return to capital r is given by

$$r = f'(k), \qquad (5)$$

and where the steady state capital-labor ratio k is determined by

$$sf(k) = k\,(g + \delta) \qquad (6)$$

We would like to study α as a function of g, but since k is a strictly decreasing function of g, we may as well study α as a function of the steady state value k. To this end we plug (5)-(6) into (4) and obtain

$$\alpha = \frac{kf'(k)}{f(k)}. \qquad (7)$$

A closer look at α ct'd

We are interested in the behavior of

$$\alpha = \frac{kf'(k)}{f(k)}.$$

as a function of k. This is closely related to the **elasticity of substitution**.

Elasticity of substitution

Definition: For any production function $F(K, L)$ we define the following objects.

1. The **marginal rate of technical substitution** M, is defined by
$$M = \frac{F_L(K, L)}{F_K(K, L)}. \tag{8}$$

2. The **elasticity of substitution** σ is, with $k = K/L$, defined by
$$\sigma = \frac{dk}{dM} \cdot \frac{M}{k} \tag{9}$$

We recall that, in market equilibrium, M is the relative price of capital w.r.t. labor, so σ measures how the relative price reacts to a change in the capital-labor ratio.

Main result for α

Proposition: With assumptions as above, the following hold

1. If $\sigma = 1$ then α is constant as a function of g.
2. If $\sigma > 1$ then α is decreasing in g.
3. If $\sigma < 1$ then α is increasing in g.

There is a considerable debate and disagreement on the empirical value of σ.

4.

The Piketty growth model.

The model

The main difference between the Piketty model and the standard model is that the standard model is dealing with **gross** quantities, whereas Piketty models **net** quantities. We thus introduce the following objects.

$$
\begin{aligned}
\widehat{F} &= \text{net production function,} \\
\widehat{Y} &= \text{net income,} \\
\widehat{I} &= \text{net investment,} \\
\widehat{s} &= \text{net savings ratio,}
\end{aligned}
$$

These quantities are related to the standard counterparts by the relations

$$
\begin{aligned}
\widehat{F}(K, L) &= F(K, L) - \delta K, \\
\widehat{Y} &= Y - \delta K, \\
\widehat{I} &= I - \delta K, \\
\widehat{I} &= \widehat{s}\widehat{Y}
\end{aligned}
$$

Assumptions

Piketty assumes the following.

- The **net production function** $\widehat{F}$ is increasing, concave, satisfies the ususal Inada conditions, and has constant returns to scale.

- The **net investment** is a constant fraction $\widehat{s}$ of net income, so

$$I - \delta K = \widehat{s}\,(Y - \delta K)\,.$$

- Labor grows according to

$$\dot{L}_t = gL_t.$$

Doing basically the same calculations as before, we obtain the main equilibrium equation as

$$\dot{k}_t = \widehat{s}\widehat{f}(k_t) - gk_t$$

Main result for the Piketty model

The ODE

$$\dot{k}_t = \widehat{s}\widehat{f}(k_t) - gk_t$$

is stable and the following will hold.

- The capital-labor ratio k_t will converge to the equilibrium value k^* which the unique solution of the equilibrium equation

$$\widehat{s}\widehat{f}(k^*) = gk^*.$$

- The capital-net-income ratio $\widehat{\beta} = K/\widehat{Y}$ will converge to the equilibrium value

$$\widehat{\beta}^* = \frac{\widehat{s}}{g}$$

  Compare with

$$\beta^* = \frac{s}{g + \delta}$$

Comparing Piketty and Solow

Proposition: Assume the Piketty model with a constant net savings rate $\widehat{s}$. Then the Solowian gross savings rate s turns out to be constant, and it is given by

$$s = \frac{\widehat{s}\,(g + \delta)}{g + \widehat{s}\delta}$$

If, in particular, we set $g = 0$ we obtain

$$s = 1$$

i.e. all income is invested, and nothing is consumed. This is not realistic.

Savings rates ct'd

Assume the standard Solow model with a constant gross savings rate s. Then the implied net savings rate $\widehat{s}$ turns out to be constant, and it is given by

$$\widehat{s} = \frac{gs}{g + \delta(1 - s)}$$

The implication of this formula for the zero growth case is that $g = 0$ implies that the net savings rate $\widehat{s}$ is zero, except for the not very interesting case when $s = 1$. This seems to be a quite reasonable implication, as opposed to the corresponding implication on the previous slide.

Capital-output ratios

Proposition: In the standard model, the steady state gross and net capital-output rates are given by

$$
\begin{aligned}
\beta &= \frac{s}{g+\delta}, \\
\widehat{\beta} &= \frac{s}{g+\delta(1-s)}.
\end{aligned}
$$

In the Piketty model, the steady state gross and net capital-output rates are given by

$$
\begin{aligned}
\beta &= \frac{\widehat{s}}{g+\delta\widehat{s}}, \\
\widehat{\beta} &= \frac{\widehat{s}}{g}.
\end{aligned}
$$

Comparing models - conclusion

Solow seems to be doing a better job than Piketty on the theoretical level.

References

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SOLOW, R. A contribution to the theory of economic growth. *Quarterly Journal of Economics 70* (1956), 65–94.

KRUSELL, P., AND SMITH, T. Is Piketty's "Second law of capitalism" fundamental? Working paper $http://aida.wss.yale.edu/smith/piketty1.pdf$, (2014).

- The discussion of the Solow model is based on Solow (1956) and can be found in any textbook.
- The discussion of the Piketty model is a continuous time version of Krusell-Smith.
- I have no claim whatsoever of originality.