Piketty for the Pedestrian

Tomas Björk
Department of Finance
Stockholm School of Economics
tomas.bjork@hhs.se

November 1:st, 2014

Very preliminary and incomplete

Abstract

These are merely my personal notes on some parts of the Piketty book [2]. The notes are written mainly in order to explain the theory to myself and to my old friend Lasse. Sections 2-3 is standard texbook material, whereas sections 4-5 draw heavily on the recent paper [1].

1 Introduction

In these notes I try to highlight some theoretical aspects of the Piketty book [2]. More precisely I discuss, in some detail, the "First and Second Laws of Capitalism" proposed in [2]. In Section 2, I do this by using a very simple kindergarten model, which is also discussed critically. Section 3 is a bit more ambitious in the sense that we study Piketty's Second Law within the framework of a standard Solow growth model in continuous time. In Section 4 we study the Piketty alternative version of the Solow model and we compare with the Solow results, very much along the lines of [1]. My treatment of the Solow model in Section 3 is standard textbook material, and my discussion of the alternative Piketty model in Sections 4-5 is merely a continuous time version of the corresponding discussion of the discrete time model in the recent paper [1]. I thus have no claims whatsoever of originality.

2 The basic Piketty theory

In his book, Piketty has basically no mathematical model for the economy. This section is an attempt to formulate the simplest possible model which allows us to derive the main results of Piketty, namely the "First and Second Fundamental Laws of Capitalism"

2.1 Notation and the First Law

We use the the following notation with the dimensions in brackets. Time is denoted by $[t]$, and money by $[m]$. The dimension of a dimensionless entity is denoted by [1].

$$\begin{aligned}
K &= \text{capital } [m] \\
Y &= \text{income per unit time } [m/t] \\
s &= \text{savings rate } [1] \\
g &= \text{growth rate of income } [1/t] \\
r &= \text{return rate on capital } [1/t] \\
Y^K &= \text{capital income per unit time } [m/t] \\
\alpha &= \text{capital income as a share of total income } [1] \\
\beta &= \text{capital-income ratio } [t]
\end{aligned}$$

By definition these objects are related as follows

$$\begin{aligned}
\beta &= \frac{K}{Y}, \\
Y^K &= rK, \\
\alpha &= \frac{Y^K}{Y}.
\end{aligned}$$

We can now immediately obtain Piketty's "First Law of Capitalism", which is little more than a definition.

Proposition 2.1 (First Fundamental Law of Capitalism) *The capital share of income is given by*

$$\alpha = \frac{rK}{Y} = r\beta.$$

Remark 2.1 *Since capital income Y^K is included in gross income Y, we must of course have $Y^K \leq Y$, implying that $\alpha \leq 1$. This implies in particular that the parameters above cannot be chosen freely. The relevant restrictions on parameters can only be determined within a full fledged equilibrium model. See Section 3 below for details.*

2.2 A dynamic model and the Second Law

As we saw above, Piketty's First Law is more or less a definition. Piketty's Second Law, on the other hand, is not a mere definition. It is a (widely debated) *result* and in order to derive it we need to introduce a mathematical model. The simplest model is probably given by the following construction.

Definition 2.1 *We define the dynamics of the capital stock K and income Y as*

$$\dot{K}_t = sY_t, \tag{1}$$

$$\dot{Y}_t = gY_t. \tag{2}$$

We are now ready to derive the second law. From the definition of β we have

$$\dot{\beta}_t = \frac{Y_t \dot{K}_t - K_t \dot{Y}_t}{Y_t^2}$$

Using the dynamics (1)-(2) we easily obtain

$$\dot{\beta}_t = s - g\beta_t. \tag{3}$$

This is a stable ODE, and we immediately have the following result.

Proposition 2.2 (Second Fundamental Law of Capitalism) *Assume that the savings rate s and the growth rate g are constant over time. Then the capital-income ratio $\beta_t = K_t/Y_t$ converges to an equilibrium value given by*

$$\beta = \frac{s}{g}.$$

2.3 Economic and political conclusions

From the Second Law and the First Law, i.e. the definition of the capital share of income as $\alpha = r\beta$, we have the following result.

Proposition 2.3 (Capital Share of Income) *In steady state, the capital share of income, α, is given by*

$$\alpha = \frac{sr}{g}.$$

Piketty's argument is now roughly as follows.

- From econometric data it seems that the return on capital is greater than the growth rate.
- This implies that capital accumulates at a faster rate than the growth rate of income, so the income share due to capital will increase.
- We see indeed in Proposition 2.3 that, by doing comparative statics, a high ratio r/g will imply a high capital share of income α.
- In particular we see that if the growth rate approaches zero, then the capital share of income may explode to the limit $\alpha = 1$.
- The politically hot implication of this is that we will have increasing income inequality between capital and labor.
- It is in fact obvious how to interpret the Piketty conclusion in Marxist class terms. According to that interpretation, and with the extreme value $\alpha = 1$, the capitalist class will in steady state get more or less all income, while the working class will get nothing above some minimum subsistence level.

As a numerical example let us set $g = 3\%$, $s = 20\%$, and $r = 5\%$ which are reasonable values for Western Europe. The steady state capital-income ratio is then given by $\beta = 6.6$ and the capital share of income is $\alpha = 33\%$. If we now assume that the growth rate decreases to 0.5% we have a change of the capital-income ratio from 6.6 to 40 and the capital share of income is increased from 33% to 80%. This is indeed quite dramatic.

2.4 Counter arguments

The results and implications of the previous section have all been widely and hotly debated. The most common counter arguments center around one or several of the following topics.

- The distribution of capital in Western society.
- Capital depreciation.
- The assumptions of constant savings rate, growth rate, and return on capital.

2.4.1 Distribution of capital

One counterargument to a Marxist interpretation goes as follows.

- In modern society capital is owned, not only by traditional "capitalists", but also to a large extent by the state and by "ordinary people". A wage earner in western society will have some capital invested in funds (pension funds and others) and he/she may also have capital invested in real estate.
- Since capital income thus will be distributed among the entire population, it is not at all clear that we can equate a large α with large income inequality.

2.4.2 Capital depreciation

Another very obvious counter argument is that Piketty has not taken *capital depreciation* into account. We thus define

$$\delta = \text{capital depreciation rate } [1/t],$$

which implies that the capital growth dynamics are now slightly changed into

$$\dot{K}_t = sY_t - \delta K_t.$$

A simple calculation gives us

$$\dot{\beta}_t = s - (g + \delta)\,\beta_t,$$

and we have the following result.

Proposition 2.4 *With a capital depreciation rate δ the steady state value of the capital income ratio β is given by*

$$\beta = \frac{s}{g+\delta}$$

and the equilibrium capital share of income is given by

$$\alpha = s\frac{r}{g+\delta}.$$

We see that the introduction of the depreciation rate δ has a profound effect on the Piketty result: Even if the growth rate g goes to zero, the capital share of income, α, will **not** explode.

As a numerical example we set $g = 2\%$, $s = 30\%$, $\delta = 5\%$ and $r = 5\%$. This gives us a capital-income ratio of $\beta = 4.2$ and the capital share of income is $\alpha = 21\%$. If we now decrease g drastically to 0% we obtain $\beta = 6$ and $\alpha = 30\%$ so the increase in β and α are not very dramatic, compared to the case studied in Section 2.3 where we had no capital depreciation.

2.4.3 Time and state varying parameters

In our derivation of the Second Law we have assumed that all parameters are constant. This is, however, note quite reasonable from an economic point of view. According to the Piketty argument, the capital stock K is likely to grow faster than income Y. It would then seem natural, from a microeconomic point of view, to conclude that the return of capital will decrease, so we would expect r do decrease as K increases. Recalling the equilibrium relation $\alpha = (sr)/g$ we see that a small value of g would lead us to expect a small value of r, so the comparative statics is misleading.

In the same way one can argue that the savings rate s is not independent of g. In particular one may argue that if g goes to zero so we have no growth, then the savings rate also goes to zero, so again we cannot do naive comparative statics.

2.4.4 Summing up the critique

We have seen, at various points, that the simple calculations in Section 2 are subject to criticism. We recall some of the critical arguments

- By definition we must have $Y^K \leq Y$, and $\alpha \leq 1$, but there is in fact no formal/mathematical restriction of that form in the equations above.
- It is thus obvious that (at least some of) the parameters s, g, and r should be interdependent and in fact dependent on Y and K.
- In particular we suspect that r varies inversely with K.

These issues can only be resolved within a more complete model than the simple one developed in Section 2. The next section is devoted to the development of such a model.

3 The Solow growth model

In this section we present a standard continuous time Solow growth model as introduced in [3]. All results can be found in any standard textbook on the subject.

3.1 Model assumptions and basic results

We introduce the following new notation, where $[p]$ denotes "person".

$$\begin{aligned} I_t &= \text{investment rate } [m/t] \\ L_t &= \text{size of labor force } [p]. \end{aligned}$$

The main player in the model is the **production function** $F : R^2 \rightarrow R$. The interpretation is that for a given capital stock K and labor force L, we produce an income rate given by $Y = F(K, L)$.

We also need some assumptions concerning the production function, savings, and the evolution of the labor force.

Assumption 3.1 *We assume the following.*

- *The production function exhibits constant returns to scale, i.e.*

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

for all positive K, L, and λ. We also assume that F is increasing and concave in K with $F(0, L) = 0$ for all K, and we assume that F satisfies the usual Inada conditions.

$$\frac{\partial F}{\partial K}(0, L) = +\infty, \qquad \lim_{x \rightarrow \infty} \frac{\partial F}{\partial K}(x, L) = 0.$$

- *Investment (in new capital) is a constant fraction of output, so that*

$$I = sF(K, L)$$

where s is the (constant) savings rate.

- *The capital depreciation rate δ is constant over time.*
- *Labor grows exponentially at the growth rate g, so*

$$L_t = e^{gt} L_0.$$

Putting all this together we obtain the following equations for the evolution of capital K, gross income Y, and labor L.

$$\dot{K}_t = sF(K_t, L_t) - \delta K_t, \quad (4)$$

$$Y_t = F(K_t, L_t), \quad (5)$$

$$\dot{L}_t = gL_t. \quad (6)$$

In order to study this system we introduce the capital-labor ratio k and the income-labor ratio y by

$$k_t = \frac{K_t}{L_t}, \quad y_t = \frac{Y_t}{L_t}.$$

We then obtain

$$\dot{k}_t = sf(k_t) - (\delta + g)\, k_t, \tag{7}$$
$$y_t = f(k_t), \tag{8}$$

where f is defined by

$$f(x) = F(x, 1).$$

The Inada conditions on F imply that f is concave and increasing with $f(0) = 0$, and that we have

$$f'(0) = +\infty \quad \lim_{x \to \infty} f'(x) = 0.$$

From these conditions we immediately see that the ODE for k is stable, and we have the classical Solow result.

Proposition 3.1 *The ODE (7) for the capital-labor ratio k is stable, and the following will hold.*

- *The capital-labor ratio k_t will converge to the equilibrium value k^* which the unique solution of the equilibrium equation*

$$sf(k^*) = (\delta + g)\, k^*. \tag{9}$$

- *The income-labor ratio will converge to the equilibrium value*

$$y^* = f(k^*). \tag{10}$$

- *At equilibrium, income and capital with grow at the rate g, i.e.*

$$\dot{Y}_t = gY_t,$$
$$\dot{K}_t = gK_t.$$

- *The capital-income ratio $\beta = K/Y$ will converge to the equilibrium value*

$$\beta^* = \frac{s}{g + \delta} \tag{11}$$

Proof. Most of the statements are obvious. To prove the last last statement we note that $K/Y = k/y$, and the equilibrium value for k/y is easily obtained as $s/(g + \delta)$ by using equations (9)-(10). ∎

3.2 Labor and capital incomes

We now go on to study the labor wage, i.e. the price of labor, w^L, and the price of kapital, w^K. According to standard microeconomic theory we should have

$$\begin{aligned} w_t^K &= F_K(K_t, L_t), \\ w_t^L &= F_L(K_t, L_t). \end{aligned}$$

The labor income per unit time, Y^L, and the capital income per unit time, Y^K, are thus given by

$$Y_t^L = L_t F_L(K_t, L_t), \tag{12}$$

$$Y_t^K = K_t F_K(K_t, L_t), \tag{13}$$

and we note (with some satisfaction) that by the Euler formula $F = KF_K + LF_L$ for linearly homogeneous functions we have

$$Y_t = Y_t^L + Y_t^K.$$

The return on capital, r, is defined as

$$r_t = \frac{Y_t^K}{K_t}$$

so from (13) we have

$$r_t = F_K(K_t, L_t).$$

Since F is homogeneous we also have

$$F_K(K, L) = F_K(k, 1) \tag{14}$$

with $k = K/L$ as above, so we obtain

$$r_t = f'(k_t), \tag{15}$$

where, as before, $f(k) = F(k, 1)$.

To compute the capital share of income, α, we have by definition

$$\alpha_t = \frac{Y_t^K}{Y_t}$$

so from (13) and (14) we obtain

$$\alpha_t = \frac{K_t}{Y_t} F_K(K_t, L_t) = \frac{K_t}{Y_t} f'(k_t) = \frac{K_t}{Y_t} r_t. \tag{16}$$

We now have the following result.

Proposition 3.2 *In steady state the capital share of income is constant and given by*

$$\alpha^* = \frac{r^* s}{g + \delta}, \tag{17}$$

where the steady state return of capital is given by

$$r^* = f'(k^*). \tag{18}$$

Proof. From Proposition 3.1 we know that K/Y converges to $\beta^* = s/(g+\delta)$. The result now follows from (16), and (15), evaluated at the steady state value k^*. ∎

From this result we see that we are really not allowed to do the naive comparative statics of Section 2.3. In that section we assumed that $\delta = 0$ so we had

$$\alpha = \frac{rs}{g}$$

and we then noted that if g decreases towards zero, then α explodes. We now see that the problem with that naive argument is that the rate of return on capital r^* will in fact depend on g. More precisely, we recall that r^* is given by

$$r^* = f'(k^*).$$

where k^* is determined by the steady state equation (9). If we now assume that $\delta = 0$ we have the equilibrium equation

$$sf(k^*) = gk^*,$$

From this we see (draw a figure) that when g decreases, the equilibrium capital-labor ratio k^* will increase. Since F, and thus f, is assumed to be concave this implies that the marginal productivity of capital f' will decrease, so r^* will also decrease. It is thus not at all clear what will happen to the ratio r/g, and in the general case nothing can be said - the quotient r/g may increase, decrease, or stay constant. The determining factor here is the *elasticity of substitution* σ. This elasticity measures the (percentage) response in the relative price of capital to labor w.r.t. a (percentage) change in the capital-labor ratio k, and in Section 2.3 we will study this in some detail. At the moment we only give an easy example of what may happen.

Example 3.1 As a concrete example let us assume that F is Cobb-Douglas, so

$$F(K, L) = K^\rho L^{1-\rho},$$

for $0 < \rho < 1$. We thus have $f(k) = k^\rho$ so the equilibrium equation is

$$sk^\rho = gk$$

and we obtain

$$k^* = \left(\frac{s}{g}\right)^{\frac{1}{1-\rho}}.$$

From $r^* = f'(k^*)$ we obtain

$$r^* = \rho\frac{g}{s}$$

and we obtain

$$\alpha^* = \frac{r^* s}{g} = \rho$$

regardless of the value of g. In the next section we will study this in more detail.

3.3 A closer look at the capital share of income

The object of this section is to take a closer look at the capital share of income as we vary g. To this end we work in steady state and recall that

$$\alpha = \frac{rs}{g+\delta} \tag{19}$$

where the return to capital r is given by

$$r = f'(k), \tag{20}$$

and where the steady state capital-labor ratio k is determined by

$$sf(k) = k\,(g+\delta) \tag{21}$$

We would like to study α as a function of g, but since k is a strictly decreasing function of g, we may as well study α as a function of the steady state value k. To this end we plug (20)-(21) into (19) and obtain

$$\alpha = \frac{kf'(k)}{f(k)}. \tag{22}$$

We are primarily interested in $\frac{d\alpha}{dk}$ but before going on to study this object we recall the following definitions from microeconomics. See Appendix A for more details on elasticity.

Definition 3.1 *For any production function $F(K,L)$ we define the following objects.*

1. *The* **marginal rate of technical substitution** *M, is defined by*

$$M = \frac{F_L(K,L)}{F_K(K,L)}. \tag{23}$$

2. *The* **elasticity of substitution** *σ is, with $k = K/L$, defined by*

$$\sigma = \frac{dk}{dM} \cdot \frac{M}{k} \tag{24}$$

We recall that, in market equilibrium, M is the relative price of capital w.r.t. labor, so σ measures how the relative price reacts to a change in the capital-labor ratio. We also recall the following easy standard result.

Proposition 3.3 *Assume that the production function F has constant returns to scale. Then the elasticity of substitution is given by the expression*

$$\sigma = \frac{f'(k)\,\{kf'(k) - f(k)\}}{kf(k)f''(k)} \tag{25}$$

We now go on to our main task, which is to study the capital-output ratio α as a function of k. Differentiating α in (22) gives us the following result.

Proposition 3.4 *In terms of comparative statics we have*

$$\frac{d\alpha}{dk} = \frac{[-f''(k)]\,k}{f(k)}\,(\sigma - 1)\,. \tag{26}$$

Since f is concave and the steady state value k is decreasing in g we have the following easy result.

Proposition 3.5 *With assumptions as above, the following hold*

1. *If $\sigma = 1$ then α is constant as a function of g.*
2. *If $\sigma > 1$ then α is decreasing in g.*
3. *If $\sigma < 1$ then α is increasing in g.*

Remark 3.1 *In Appendix A the reader is given more details concerning the elasticity concept and it's properties. See in particular Proposition A.2.*

For a Cobb-Douglas production function we have $\sigma = 1$ so, as we could see already in the previous section, α is constant.

Going back to the original Piketty situation, we assume that $\delta = 0$ and study α as $g \to 0$. With $\delta = 0$ this implies that $k \to +\infty$ so we need to study the limit (if it exists)

$$\alpha = \frac{kf'(k)}{f(k)}.$$

If we assume that F is CES, i.e. F has constant elasticity of substitution, then F has the form

$$F(K, L) = \left[\gamma K^{-\rho} + (1-\gamma)L^{-\rho}\right]^{-\frac{1}{\rho}}, \quad \rho > -1$$

It is an easy exercise to show that the elasticity of substitution, σ, is given by

$$\sigma = \frac{1}{1+\rho}$$

After some computations we have the following result.

Proposition 3.6 *Assume that F is CES as above. Then the following hold.*

1. *For the case $\sigma > 1$, we have*

$$\lim_{g\to 0} \alpha = \lim_{k\to\infty} \alpha = 1.$$

2. *For the case $\sigma < 1$, we have*

$$\lim_{g\to 0} \alpha = \lim_{k\to\infty} \alpha = 0.$$

In other words: For the CES case, the Piketty explosion of capital's share of income occurs if and only if $\sigma > 1$. For the case $\sigma <$ we instead have an complete collapse of capital's share of income. A lot of the debate around Piketty's book has thus centered around the size of σ.

4 The Piketty alternative to the Solow model

In this section we present the Piketty version of the standard Solow model. This alternative model is never stated clearly in the book [2], but rather derived from verbal arguments in the book and from earlier writings of Piketty. This entire section is merely a continuous time version of the model presented in [1].

4.1 The model

The main difference between the Piketty model and the standard model is that the standard model is dealing with *gross* quantities, whereas Piketty models *net* quantities. We thus introduce the following objects.

$$\begin{aligned} \widetilde{F} &= \text{net production function,} \\ \widetilde{Y} &= \text{net income,} \\ \widetilde{I} &= \text{net investment,} \\ \widetilde{s} &= \text{net savings ratio,} \end{aligned}$$

These quantities are related to the standard counterparts by the relations

$$\begin{aligned} \widetilde{F}(K,L) &= F(K,L) - \delta K, \\ \widetilde{Y} &= Y - \delta K, \\ \widetilde{I} &= I - \delta K, \\ \widetilde{I} &= \widetilde{s}\widetilde{Y} \end{aligned}$$

The assumptions are as follows.

Assumption 4.1 *We assume the following.*

- *The net production function* $\widetilde{F}$ *is increasing, concave, satisfies the ususal Inada conditions, and has constant returns to scale.*
- *The net investment is a constant fraction* $\widetilde{s}$ *of net income, so*

$$I - \delta K = \widetilde{s}\,(Y - \delta K)\,.$$

- *Labor grows according to*

$$\dot{L}_t = gL_t.$$

We note that

- Capital, K, and labor, L, are the same in the standard and in the Piketty model. All other objects are changed.
- With the Piketty assumptions, the gross production function F can **not** satisfy the Inada conditions. In particular we must have $F_K(K,L) \geq \delta$.
- With the Piketty assumptions, the gross production function F can **not** have constant returns to scale, i.e. F is not linearly homogeneous.

Very much as in the standard model we now have the following relations

$$\begin{aligned}\widetilde{Y}_t &= \widetilde{F}(K_t, L_t),\\ \widetilde{I}_t &= \widetilde{s}\widetilde{Y}_t,\\ \dot{K}_t &= \widetilde{I}_t\end{aligned}$$

We thus obtain the following dynamics for capital.

$$\dot{K}_t = \widetilde{s}\widetilde{F}(K_t, L_t) \tag{27}$$

Defining the capital-labor ration k as usual by

$$k = K/L$$

and the scaled net production function $\widetilde{f}$ by

$$\widetilde{f}(k) = \widetilde{F}(k, 1),$$

we obtain

$$\dot{k}_t = \widetilde{s}\widetilde{f}(k_t) - gk_t \tag{28}$$

We now have the following result, which is parallel to Proposition 3.1.

Proposition 4.1 *The ODE (28) for the capital-labor ratio k is stable, and the following will hold.*

- *The capital-labor ratio k_t will converge to the equilibrium value k^* which the unique solution of the equilibrium equation*

$$\widetilde{s}\widetilde{f}(k^*) = gk^*. \tag{29}$$

- *Defining the net income-labor ratio $\widetilde{y}$ by $\widetilde{y} = \widetilde{Y}/L$ we have*

$$\widetilde{y}_t = \widetilde{f}(k_t) \tag{30}$$

and $\widetilde{y}_t$ will converge to the equilibrium value

$$\widetilde{y}^* = \widetilde{f}(k^*). \tag{31}$$

- *In steady state, net income and capital with grow at the rate g, i.e.*

$$\begin{aligned}\dot{\widetilde{Y}}_t &= gY_t,\\ \dot{K}_t &= gK_t.\end{aligned}$$

- *The capital-net-income ratio $\widetilde{\beta} = K/\widetilde{Y}$ will converge to the equilibrium value*

$$\widetilde{\beta}^* = \frac{\widetilde{s}}{g} \tag{32}$$

5 Comparing the Piketty and the standard model

We now go on to compare the results of the Piketty model with the corresponding results in the standard Solow model.

5.1 Comparing savings rates between models

In this section we study the implied gross savings rate in the Piketty model and the implied net savings rate in the standard model.

5.1.1 The gross savings rate s in the Piketty model

The first question we attack is the following. Suppose we accept the assumptions of the Piketty model, so in particular we assume a constant net savings rate $\widetilde{s}$. What can we then say about the implied gross savings rate s? The answer is given by the following result.

Proposition 5.1 *Assume the Piketty model with a constant net savings rate $\widetilde{s}$. Then the implied gross savings rate s turns out to be constant, and it is given by*

$$s = \frac{\widetilde{s}\,(g+\delta)}{g+\widetilde{s}\delta} \tag{33}$$

Proof. We work at steady state. by definition we have $s = I/Y$ so

$$s = \frac{i}{y}$$

where $i = I/L$ and $y = Y/L$. By definition we also have $\widetilde{I} = \widetilde{s}\widetilde{Y}$ which reads

$$I - \delta K = \widetilde{s}\,(Y - \delta K)\,,$$

implying $i - \delta k = \widetilde{s}\,(y - \delta k)$. This gives us

$$i = \widetilde{s}y + (1-\widetilde{s})\,\delta k$$

On the other hand, we have from (29)-(30) and the relation $\widetilde{y} = y - \delta k$

$$\widetilde{s}\,(y - \delta k) = gk$$

which gives us

$$k = \frac{\widetilde{s}y}{g+\widetilde{s}\delta}$$

Plugging this formula for k into the expression for i above and dividing by y gives us the result. ∎

From an economic point of view this result is in fact quite disturbing. We see this if we set $g = 0$, which is exactly the case that Piketty is so focused on. From formula (33) we see that when the growth rate g is zero, the Piketty model implies that the gross savings rate in steady state is given by $s = 1$. In other words: The entire gross income is saved. This seems to be a very unrealistic conclusion indeed.

5.1.2 The net savings rate $\widetilde{s}$ in the standard model

We now reverse the question of the previous section, and accept the assumptions of the standard model, so in particular we assume a constant gross savings rate s. What can we then say about the implied net savings rate $\widetilde{s}$? The answer is given by the following result.

Proposition 5.2 *Assume the standard Solow model with a constant gross savings rate s. Then the implied net savings rate $\widetilde{s}$ turns out to be constant, and it is given by*

$$\widetilde{s} = \frac{gs}{g + \delta(1 - s)} \tag{34}$$

Proof. We work at steady state.We have

$$\widetilde{s} = \frac{I - \delta K}{Y - \delta K} = \frac{sY - \delta K}{Y - \delta K} = \frac{s - \delta\beta}{1 - \delta\beta}$$

Using (11) for the steady state capital-output ratio β gives us the result. ∎

The implication of this formula for the zero growth case is that $g = 0$ implies that the net savings rate $\widetilde{s}$ is zero, except for the not very interesting case when $s = 1$. This seems to be a quite reasonable implication, as opposed to the corresponding implication in Section 5.1.1.

5.2 Comparing capital-output ratios between models

In this section we study the implied gross capital-output ratio in the Piketty model and the implied net capital-output rate in the standard model. It is easy to derive the following result.

Proposition 5.3 *The following hold.*

1. *In the standard model, the steady state gross and net capital-output rates are given by*

$$\beta = \frac{s}{g + \delta}, \tag{35}$$

$$\widetilde{\beta} = \frac{s}{g + \delta(1 - s)}. \tag{36}$$

2. *In the Piketty model, the steady state gross and net capital-output rates are given by*

$$\beta = \frac{\widetilde{s}}{g + \delta\widetilde{s}}, \tag{37}$$

$$\widetilde{\beta} = \frac{\widetilde{s}}{g}. \tag{38}$$

Proof. For the standard model, equation (35) is merely a restatement of (11). To prove (36) we note that

$$\widetilde{\beta} = \frac{K}{Y - \delta K} = \frac{\beta}{1 - \delta\beta}$$

and then use (35).

For the Piketty model, equation (38) is just a restatement of (32). In order to prove (37) we note that

$$\frac{\widetilde{s}}{g} = \widetilde{\beta} = \frac{K}{Y - \delta K} = \frac{\beta}{1 - \delta\beta}.$$

Solving for β gives us the result. ∎

Let us assume that $\delta > 0$, since otherwise the models coincide. We then see that in the standard model, both β and $\widetilde{\beta}$ stay bounded as g goes to zero. In the Piketty model, on the other hand, the gross capital-output ratio β stays bounded whereas the net capital-output ratio goes to infinity.

A A recap on elasticity

In this appendix we give a brief recap of the concept of elasticity measures in economics. We start with the purely mathematical part.

Definition A.1 *Let $y : R \to R$ be a given real valued continuously differentiable function. The* **elasticity** *of y with respect to the variable x is denoted by σ_x^y, and it is defined by*

$$\sigma_x^y(x) = \frac{dy}{dx} \cdot \frac{x}{y}$$

In more intuitive terms we can write

$$\sigma_x^y(x) = \left(\frac{dy}{y}\right) \Big/ \left(\frac{dx}{x}\right)$$

and we see that the elasticity measures how a relative (i.e. percentage) change in x causes a relative change in y. If the function y is invertible then it is easy to see that we have

$$\sigma_y^x = \frac{1}{\sigma_x^y}.$$

In economics, the main use of the elasticity concept comes from the following very simple results.

Proposition A.1 *Consider a given non-negative function $y(x)$. Then the following hold.*

- *The function h defined by*

$$h(x) = \frac{y(x)}{x}$$

is increasing in x if and only if $\sigma > 1$. More precisely we have

$$h'(x) = \frac{y}{x^2} \left\{ \sigma_x^y(x) - 1 \right\}$$

- *The function h defined by*

$$h(x) = y(x)x$$

is increasing in x if and only if $\sigma > -1$. More precisely we have

$$h'(x) = y \left\{ \sigma_x^y(x) + 1 \right\}$$

We will now apply this result in order to study the factor shares of income for capital and labor in competitive equilibrium, and to this end we recall the following definition.

Definition A.2 *Consider a constant returns to scale production function $F(K, L)$.*

1. *The* **marginal rate of technical substitution** *M, is defined by*

$$M = \frac{F_L(K, L)}{F_K(K, L)}. \tag{39}$$

2. *The* **elasticity of substitution** *σ is, with $k = K/L$, defined by*

$$\sigma_M^k = \frac{dk}{dM} \cdot \frac{M}{k} \tag{40}$$

We now define w^K as the price of capital and w^L as the labor wage. The total capital income is then $Y^K = w^K \cdot K$ and the labor income is $Y^L = w^L \cdot L$, so the factor share of labor over capital is given by

$$\frac{Y^L}{Y^K} = \frac{w^L \cdot L}{w^K \cdot K}$$

From microeconomics we recall that in equilibrium we have

$$\frac{F_L}{F_K} = \frac{w^L}{w^K}$$

so we can write

$$\frac{Y^L}{Y^K} = \frac{F_L \cdot L}{F_K \cdot K} = \frac{M}{k}.$$

From proposition A.1 we then have the following result.

Lemma A.1 *With notation as above, the following hold.*

1. The labor share of income is increasing in k if and only if $\sigma_k^M > 1$.

2. The labor share of income is decreasing in k if and only if $\sigma_k^M < 1$.

We note that this lemma involves the elasticity σ_k^M rather than the elasticity of substitution σ_M^k from Definition A.2. In terms of the capital share of income and using the elasticity σ_M^k we then have the following result.

Proposition A.2 *Denoting the elasticity of substitution σ_M^k by σ we have the following relations.*

1. The capital share of income is increasing in k if and only if $\sigma > 1$.

2. The capital share of income is decreasing in k if and only if $\sigma < 1$.

Proof. Use the fact that $Y^K/Y^L = \left(Y^L/Y^K\right)^{-1}$, and that $\sigma = 1/\sigma_k^M$. ▮

References

[1] Krusell, P., and Smith, T. Is Piketty's "Second law of capitalism" fundamental? Working paper *http://aida.wss.yale.edu/smith/piketty1.pdf*, (2014).

[2] Piketty, T. *Capital in the Twenty-First Century.* Harward, 2014.

[3] Solow, R. A contribution to the theory of economic growth. *Quarterly Journal of Economics 70* (1956), 65–94.