

The Pedestrian's Guide to Local Time ^{*}

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Abstract

These notes contains an introduction to the theory of Brownian and diffusion local time, as well as its relations to the Tanaka Formula, the extended Ito-Tanaka formula for convex functions, the running maximum process, and the theory of regulated stochastic differential equations. The main part of the exposition is very pedestrian in the sense that there is a considerable number of intuitive arguments, including the use of the Dirac delta function, rather than formal proofs. For completeness sake we have, however, also added a section where we present the formal theory and give full proofs of the most important results. In the appendices we briefly review the necessary stochastic analysis for continuous semimartingales.

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1 Introduction

It seems to be a rather well established empirical fact that many students find it hard to get into the theory of local time. The formal definitions often seem non-intuitive and the proofs seem to be very technical. As a result, the prospective student of local time ends up by feeling rather intimidated and simply leaves the subject.

The purpose of these notes is to show that, perhaps contrary to common belief, local time is very intuitive, and that half an hour of informal reflection on the subject will be enough to make the main results quite believable.

The approach of the notes is thus that we take an intuitive view of local time, we perform some obvious calculations without bothering much about rigor, we feel completely free to use infinitesimal arguments, and in particular we use the Dirac delta function.

Arguing in this fashion, we are led to informal proofs for many of the main results concerning local time, including the relation of Brownian local time to the running maximum, the Tanaka formula, the extended Ito-Tanaka formula for convex functions, and the theory of regulated SDE:s.

The hope is that, after having read the intuitive parts of the notes in Sections 2 - 4, the reader should feel reasonably familiar with the concepts and central results of local time, and that this should be enough to make him/her motivated to study the full formal theory.

In order to retain some minimal street cred in mathematical circles I have also, in Section 5, added full formal proofs of some of the main results. In order to make the notes more self contained I have also, in the appendices, added a very brief overview of stochastic analysis for continuous semimartingales. This includes stochastic dominated convergence and the BDG inequality, as well as the Kolomogorov continuity criterion.

There is of course no claim to originality in the intuitive approach taken in the notes. This is probably the standard way of thinking for most people who are familiar with the theory.

2 Local time for deterministic functions

In this section we introduce local time for deterministic functions, and derive some of the most important properties. The reason for spending time on the rather trivial case of deterministic functions is that a number of concepts and results, which we later will study in connection with Brownian motion, are very easily understood within the simpler framework. Lebesgue measure will be denoted by either $dm(x)$ or dx .

2.1 Introduction

Let X be an arbitrary continuous function $X : R_+ \rightarrow R$, with the informal interpretation that $t \mapsto X_t$ is a trajectory of a stochastic process. We now

consider a Borel set $A \subseteq R$ and we want to study the time spent in the set A by the X -trajectory.

Definition 2.1 We define **occupation time** $T_t(A)$ as the time spent by X in the set A on the time interval $[0, t]$. Formally this reads as

$$T_t(A) = \int_0^t I_A \{X_s\} ds,$$

where I_A is the indicator function of A .

For a fixed t , it is easy to see that T_t is a Borel measure on R , and there are now (at least) two natural questions to ask about this measure.

1. Under what conditions is T_t absolutely continuous w.r.t. Lebesgue measure, i.e. when do we have $T_t(dx) \ll dx$?
2. If $T_t(dx) \ll dm(x)$, we can define L_t^x by

$$L_t^x = \frac{T_t(dx)}{m(dx)},$$

and a natural question to ask is under what conditions L is jointly continuous in (t, x) .

The function L_t above is in fact the main actor in the present text, so we write this as a separate definition.

Definition 2.2 Assume that $T_t(dx) \ll dx$. We then define the **local time** of X as the Radon-Nikodym derivative

$$L_t^x = \frac{T_t(dx)}{m(dx)}.$$

Remark 2.1 We note that the physical dimension of local time is time over distance, i.e. $(\text{velocity})^{-1}$.

The main object of this text is to study local time for Brownian motion and diffusion processes, to show that it exists and that it is indeed continuous in (t, x) , to use Brownian local time to obtain an extension of the standard Ito formula, and to discuss the relation between Brownian local time and the theory of regulated SDE:s.

2.2 Absolute continuity of T_t and continuity of L_t

In order to get some intuitive understanding of absolute continuity of $T_t(dx)$ and continuity of L_t^x we now study some simple concrete examples.

2.2.1 The case when X is constant

It is very easy to see that we cannot in general expect to have $T_t(dx) \ll dm(x)$. A trivial counterexample is given by any function which is constant on a time set of positive Lebesgue measure. Consider for example the function X defined by $X_t = a$ for all $t \geq 0$. For this choice of X we have

$$T_t(A) = t \cdot I_A(a),$$

so the measure T_t corresponds to a point mass at a . Thus T_t is obviously not absolutely continuous w.r.t Lebesgue measure.

2.2.2 The case when X is differentiable

From the previous example we see that in order to guarantee $T_t(dx) \ll dm(x)$ we cannot allow X to be constant on a time set of positive Lebesgue measure. It is therefore natural to consider the case when X strictly increasing and for simplicity we assume that X continuously differentiable. From the definition we then have

$$T_t(A) = \int_0^t I_A \{X_s\} ds.$$

We now change variable by introducing $x = X_s$, which gives us

$$\begin{aligned} s &= X^{-1}(x), \\ ds &= \frac{dx}{X'(X^{-1}(x))}. \end{aligned}$$

Using this we obtain

$$T_t(A) = \int_A \left[\frac{1}{X'(X^{-1}(x))} I \{X(0) \leq x \leq X(t)\} \right] dx.$$

From this we see that we do indeed have $T_t(dx) \ll dx$ and that the local time is given by

$$L_t^x = \frac{1}{X'(X^{-1}(x))} \cdot I \{X(0) \leq x \leq X(t)\}.$$

Because of the appearance of the indicator in the expression above we immediately see that L is not continuous in t and x . The fact that we have a term of the form $1/X'$ is more or less what could be expected from the fact that, according to Remark 2.1, local time has dimension (velocity) $^{-1}$. We also see that we would have a major problem if $X'_t = 0$ for some t , and this is very much in line with our discovery in the previous section that we cannot allow a function which is locally constant if we want to ensure the existence of local time.

We now go on to consider a more general (not necessarily increasing) function X . We assume that X is continuously differentiable and that the level sets of X are discrete. In other words we assume that, for every (t, x) , the

set $\{s \leq t : X_s = x\}$ is a finite (or empty) set. This means that we do not allow functions which are locally constant and we also rule out functions with oscillatory behavior.

Using more or less the same arguments as for the previous example it is easy to see that we now have the following result.

Proposition 2.1 *Assume that X is differentiable with discrete level sets. Then local time L_t^x exists and is given by the formula*

$$L_t^x = \sum_{\substack{X_s = x, \\ 0 \leq s \leq t}} \frac{1}{|X'_s|} \cdot I \left\{ \inf_{s \leq t} X_s \leq x \leq \sup_{s \leq t} X_s \right\}.$$

We can immediately note some properties of L for the case of a differentiable X .

1. For fixed x , the mapping $t \mapsto L_t^x$ is non-decreasing.
2. As long as $X_t \neq x$ the mapping $t \mapsto L_t^x$ is piecewise constant.
3. At a hitting time, i.e. when $X_t = x$, then $L(x)$ has a jump at t with jump size $\Delta L_t^x = 1/|X'_t|$. This is also more or less what could be expected from Remark 2.1.
4. If $x < \inf_s X_s$ or $x > \sup_s X_s$ then $L_t^x = 0$ for all t .
5. For fixed t , the mapping $x \mapsto L_t^x$ continuous except when x is a stationary point of the X trajectory in which case $L_t^x = +\infty$.

From the discussion above we see that in order to have any chance of obtaining a local time which is continuous in (t, x) we need to consider a function X which, loosely speaking, has $|X'_t| = +\infty$ at all points, and which is also rapidly oscillating. Intuitively speaking we could then hope that the finite sum above is replaced by an infinite sum with infinitesimal terms. Using more precise language we thus conjecture that the only X trajectories for which local time may exist as a continuous function are those with locally infinite variation.

This observation leads us to study local time for Brownian motion and we will indeed see that when X is Wiener process, then continuous local time exists almost surely.

2.3 An integral formula

Let us consider a function X for which local time does exist, but where we make no assumption of continuity. Recalling the definitions

$$T_t(A) = \int_0^t I_A \{X_s\} ds,$$

$$L_t^x = \frac{T_t(dx)}{dx}.$$

we obtain the formula

$$\int_0^t I_A \{X_s\} ds = \int_A L_t^x dx = \int_R I_A(x) L_t^x dx.$$

Using a standard approximation argument we have the following result which shows that we may replace a time integral with a space integral.

Proposition 2.2 *Let $f : R \rightarrow R$ be a Borel measurable function in $L^1(L_t^x dx)$. We then have*

$$\int_0^t f(X_s) ds = \int_R f(x) L_t^x dx.$$

2.4 Connections to the Dirac delta function

Local time has interesting connections with the Dirac delta function, and we now go on to discuss these.

2.4.1 The Dirac delta function

In this section we give a heuristic definition of the Dirac function and discuss some of its properties.

For a fixed $y \in R$, the **Dirac delta function** $\delta_y(x)$ can informally be viewed as a “generalized function” defined by the limit

$$\delta_y(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} I \{y - \epsilon \leq x \leq y + \epsilon\}.$$

This limit is not a function in the ordinary sense, but we can view the expression $\delta_a(x)dx$ as a unit point mass at $x = y$. We can also view δ as a distribution, in the sense of Schwartz, namely that for any continuous function $f : R \rightarrow R$ we would have

$$\int_R f(x) \delta_y(x) dx = f(y).$$

Using the limit definition above we also obtain the formulas

$$\begin{aligned} \delta_y(x) &= \delta_0(x - y), \\ \int_R f(x) \delta_x(y) dx &= f(y). \end{aligned}$$

The Dirac function is intimately connected with the Heaviside function H defined by

$$H_y(x) = \begin{cases} 0 & \text{for } x < y, \\ 1 & \text{for } x \geq y \end{cases}$$

The Heaviside function is of course not differentiable at $x = y$, but to get some feeling for what its “derivative” should be, we approximate H_y by

$$H_y^\epsilon(x) = \begin{cases} 0 & \text{for } x < y - \epsilon, \\ \frac{1}{2\epsilon}(x - [y - \epsilon]) & \text{for } y - \epsilon \leq x \leq y + \epsilon \\ 1 & \text{for } x \geq y + \epsilon \end{cases}$$

The derivative of this function is (apart from the points $y \pm \epsilon$) easily seen to be given by

$$\frac{dH_y^\epsilon}{dx}(x) = \frac{1}{2\epsilon} I \{y - \epsilon \leq x \leq y + \epsilon\}.$$

As $\epsilon \rightarrow 0$ this is exactly the definition of the Dirac function, and we have given a heuristic argument for the following result.

Proposition 2.3 *The derivative of the Heaviside function is, in the distributional sense, given by*

$$H'_y(x) = \delta_y(x).$$

2.4.2 The delta function and local time

We recall the definition of local time as

$$L_t^x = \frac{T_t(dx)}{dx},$$

where occupation time T_t was defined by

$$T_t(A) = \int_0^t I_A \{X_s\} ds.$$

We would thus expect to have

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t I_{[x-\epsilon, x+\epsilon]} \{X_s\} ds.$$

taking the limit inside the integral we thus expect to have the following result.

Proposition 2.4 *We may, at least informally, view local time as given by the expression*

$$L_t^x = \int_0^t \delta_x(X_s) ds,$$

or on differential form

$$dL_t^x = \delta_x(X_t) dt.$$

3 Local time for Brownian motion

We now go on to study local time for a Wiener process W .

3.1 Basic properties of Brownian local time

Based on the irregular behavior of the Wiener trajectory and the considerations of Section 2 we expect occupation time to be absolutely continuous w.r.t. Lebesgue measure. Based on the informal arguments of Section 2.2.2 we also have some hope that, L_t^x is continuous in (t, x) . That this is indeed the case is guaranteed by a very deep result due to Trotter. We state the Trotter result as a part of the following theorem where we also collect some results which were proved already in Section 2.

Theorem 3.1 *For Brownian motion W , local time L_t^x does exist and the following hold.*

1. *For almost every ω , the function $L_t^x(\omega)$ is jointly continuous in (t, x) .*
2. *L_t^x can be expressed as the limit*

$$L_t^x = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t I \{x - \epsilon \leq W_s \leq x + \epsilon\} ds.$$

3. *For every fixed x , the process $t \mapsto L_t^x$ is nonnegative, nondecreasing, and increases only on the set $\{t \geq 0 : W_t = x\}$.*
4. *Informally we can express local time as*

$$L_t^x = \int_0^t \delta_x(W_s) ds,$$

or on differential form

$$dL_t^x = \delta_x(W_t) dt.$$

5. *For every bounded measurable function $f : R \rightarrow R$ we have*

$$\int_0^t f(W_s) ds = \int_R f(x) L_t^x dx.$$

From item 3 of this result it is clear that the process $t \mapsto L_t^x$ is a rather strange animal. Firstly we note that for any fixed x , the corresponding level set $D_x = \{t \geq 0 : W_t = x\}$ has Lebesgue measure zero. This follows directly from the fact that occupation time $T_t(x)$ is absolutely continuous w.r.t. Lebesgue measure, but we can also prove it directly by noting that we have $m(D_x) = \int_0^t I \{W_s = x\} ds$, so we have $E[D_x] = \int_0^t P(W_s = x) ds = 0$. Since $m(D_x) \geq 0$ we thus have $m(D_x) = 0$ almost surely. One can in fact prove the following much stronger result

Proposition 3.1 *For a fixed x , the level set $D_x = \{t \geq 0 : W_t = x\}$ is a closed set of Lebesgue measure zero with the property that every point in D_x is an accumulation point.*

The process L_t^x is thus a non decreasing process with continuous trajectories which increases on a set of Lebesgue measure zero. It is very hard to imagine what such a trajectory looks like, but one may for example think of the Cantor function.

We end this section by computing the expected value of local time.

Proposition 3.2 *Denoting the density of W_t by $p(t, x)$ we have*

$$E[L_t^x] = \int_0^t p(s, x) ds.$$

Proof. Using item 2 of the theorem above, and denoting the cdf of W_t by $F(t, x)$, we have

$$\begin{aligned}
E[L_t^x] &= E\left[\lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t I\{x - \epsilon \leq W_s \leq x + \epsilon\} ds\right] \\
&= \int_0^t \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} P(x - \epsilon \leq W_s \leq x + \epsilon) ds \\
&= \int_0^t \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \{F(s, x + \epsilon) - F(s, x - \epsilon)\} ds \\
&= \int_0^t p(s, x) ds. \blacksquare
\end{aligned}$$

3.2 Local time and Tanaka's formula

Let W again be a Wiener process and consider the process $f(W_t)$ where $f(x) = |x|$. We would now like to compute the stochastic differential of this process, and a formal application of the standard Ito formula gives us

$$df(W_t) = f'(W_t)dW_s + \frac{1}{2}f''(W_t)dt.$$

This formula is of course not quite correct, because of the singularities of the first and second derivatives of f at $x = 0$. Nevertheless it is tempting to interpret the derivatives in the distributional sense. We would then have

$$f'(x) = \begin{cases} -1 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0 \end{cases}$$

so f' is a slightly modified Heaviside function. Arguing as in Section 2.4.1 we thus have

$$f'(x) = 2\delta_0(x).$$

We now formally insert these expressions for f' and f'' into the Ito formula above and integrate. We then obtain the expression

$$|W_t| = \int_0^t \operatorname{sgn}(W_s)dW_s + \int_0^t \delta_0(W_s)ds,$$

where sgn denotes the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0 \end{cases}$$

From Theorem 3.1 we know that $L_t^x = \int_0^t \delta_x(W_s)ds$ so we would formally have the relation

$$|W_t| = \int_0^t \operatorname{sgn}(W_s)dW_s + L_t^0.$$

It is easy to extend this argument to the case when $f(x) = |x - a|$ for some real number a . Arguing as above we have then given a heuristic argument for the following result.

Theorem 3.2 (Tanaka's formula) *For any real number a we have*

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

The full formal proof of the Tanaka result is not very difficult. The idea is simply to approximate $f(x) = |x|$ with a sequence $\{f_n\}$ of smooth functions such that $f_n \rightarrow f$, apply the standard Ito formula to each f_n , and go to the limit.

We end this section by connecting the Tanaka formula with the Doob-Meyer decomposition. We then observe that since $W_t - a$ is a martingale and the mapping $x \mapsto |x|$ is convex, the process $|W_t - a|$ is a submartingale. According to the Doob-Meyer Theorem we can thus decompose $|W_t - a|$ uniquely as

$$|W_t - a| = M_t^a + A_t^a$$

where for each a , the process M^a is a martingale and A^a is a nondecreasing process. Comparing to the Tanaka formula we can thus make the identification $M_t^a = \int_0^t \operatorname{sgn}(W_s - a) dW_s$ and $A_t^a = L_t^a$.

3.3 An alternative definition of local time

We may in fact use the Tanaka formula to obtain an alternative definition of Brownian local time.

Definition 3.1 *For any real number a , we define local time by*

$$L_t^a = |W_t - a| - |a| - \int_0^t \operatorname{sgn}(W_s - a) dW_s$$

The advantage of this definition is of course that the existence of local time presents no problem. Continuity in t is obvious, but the hard work is to show continuity in (t, a) , and also to show that we have the property (which previously was a definition)

$$T_t(E) = \int_E L_t^a da,$$

for all Borel sets E .

3.4 Regulated Brownian motion

We now turn to the concept of **regulated Brownian motion**. This is a topic which is interesting in its own right, and it is also closely connected to Brownian local time and to the distribution of the Brownian running maximum.

3.5 The Skorohod construction

We start by the following result by Skorohod, which is stated for an arbitrary continuous function X . We would now like to “regulate” X in such a way that X stays positive, and we would also like this regulation to be done in some minimal way. The following result by Skorohod gives a precise formulation and solution to the intuitive problem.

Proposition 3.3 (Skorohod) *Let X be an arbitrary continuous function with $X_0 = 0$. Then there exists a unique pair (Z, F) such that*

- $Z_t = X_t + F_t$, for all $t \geq 0$,
- $Z_t \geq 0$, for all $t \geq 0$.
- F is nondecreasing with $F_0 = 0$.
- F increases only when $Z_t = 0$.

Proof. We see immediately that if a pair (Z, F) exists, then $F_t \geq X_t^-$ for all t , and since F is nondecreasing we must in fact have $F_t \geq \sup_{s \leq t} X_s^-$. This leads us to conjecture that in fact $F_t = \sup_{s \leq t} X_s^-$, and it is easy to see that with this choice of F all the requirements of the proposition are satisfied. To prove uniqueness, let us assume that there are two solutions (Z, F) and (Y, G) to the Skorohod problem so that

$$\begin{aligned} Z_t &= X_t + F_t, \\ Y_t &= X_t + G_t. \end{aligned}$$

Define ξ by $\xi_t = Z_t - Y_t = F_t - G_t$. Using the fact that ξ is continuous and of bounded variation then have

$$d\xi_t^2 = 2\xi_t d\xi_t = 2(Z_t - Y_t)(dF_t - dG_t).$$

Since F is constant off the set $\{Z_t = 0\}$ we have $Z_t dF_t = 0$ and in the same way we have $Y_t dG_t = 0$. We thus obtain

$$\xi_t^2 = -2 \int_0^t (Z_s dG_s + Y_s dF_s) \leq 0.$$

but since ξ^2 obviously is positive we conclude that $\xi = 0$. ■

We note that condition that F increases only on the set $\{Z_t = 0\}$ formalizes the idea that we are exerting a minimal amount of regulation.

3.6 Local time and the running maximum of W

In this section we will investigate the relations between local time, regulated Brownian motion, and the running maximum of a Wiener process. We start by recalling the Tanaka formula

$$|W_t| = \int_0^t \operatorname{sgn}(W_s) dW_s + L_t^0.$$

where $L^0 = L^0(W)$ is local time at zero for W . We note that the process M defined by

$$M_t = \int_0^t \operatorname{sgn}(W_s) dW_s,$$

is a martingale and, since we obviously have $(dM_t)^2 = dt$, it follows from the Levy characterization (see Corollary A.1) that M is in fact a Wiener process, so we can write $M = \tilde{W}$. We may thus write the Tanaka formula as

$$|W_t| = \tilde{W}_t + L_t^0.$$

We obviously have $|W_t| \geq 0$ and we recall that L^0 only increases on the set $\{W_t = 0\}$ i.e. on the set $\{|W_t| = 0\}$. Let us now apply the Skorohod regulator to the process \tilde{W} . We then obtain

$$Z_t = \tilde{W}_t + F_t,$$

where $Z \geq 0$, and F is increasing only on $\{Z_t = 0\}$. Comparing these two expressions we see that the pair $(|W|, L^0)$ is in fact the solution to the Skorohod regulator problem for the process \tilde{W} so we have $Z_t = |W_t|$ and $F_t = L_t^0(W)$. Furthermore, we know from the Skorohod theory that $F_t = \sup_{s \leq t} \tilde{W}_s^-$, so we obtain

$$L_t^0(W) = \sup_{s \leq t} \tilde{W}_s^-.$$

We emphasize that $L^0(W)$ is local time for W but that the supremum in the right hand side of this equality is for \tilde{W} , so we are not allowed to identify $L^0(W)$ with $\sup_{s \leq t} W_s^-$. We note however that we have

$$\tilde{W} \stackrel{d}{=} W,$$

where $\stackrel{d}{=}$ denotes equality in distribution. Using this and the symmetry of the Wiener process we obtain

$$L_t^0(W) = \sup_{s \leq t} \tilde{W}_s^- \stackrel{d}{=} \sup_{s \leq t} W_s^- \stackrel{d}{=} \sup_{s \leq t} W_s.$$

We have thus proved the following result.

Proposition 3.4 *Let W be a Wiener process and define the running maximum by*

$$S_t = \sup_{s \leq t} W_s.$$

We then have the relation

$$L_t^0(W) \stackrel{d}{=} S_t.$$

We may in fact improve this result considerably. We consider the Wiener process W , define the running maximum S as above, and consider the Skorohod regulator for the process $(-W)$. Using the Skorohod result, and recalling the Tanaka formula, we then obtain

$$|\widehat{W}_t| = -W_t + \sup_{s \leq t} [(-W_s)^-],$$

where \widehat{W} is another Wiener process. On the other hand we have

$$\sup_{s \leq t} [(-W_s)^-] = \sup_{s \leq t} (W_s) = S_t$$

so

$$|\widehat{W}_t| = -W_t + S_t.$$

On the other hand, we have the trivial equality

$$S_t - W_t = -W_t + S_t.$$

so we obtain

$$(S - W, S) = (|\widehat{W}|, S)$$

From Skorohod and the Tanaka formula we know that S is local time for \widehat{W} so $S_t = L_t^0(\widehat{W})$. This gives us

$$(S - W, S) = (|\widehat{W}|, L^0(\widehat{W}))$$

and, since W and \widehat{W} have the same distribution, we have proved the following result.

Proposition 3.5 *Let W be a Wiener process, S its running maximum, and L^0 its local time at zero. We then have the following distributional equality*

$$(S - W, S) \stackrel{d}{=} (|W|, L^0).$$

4 Local time for continuous semimartingales

In this section we extend our analysis of local time to the case of a continuous semimartingale X with decomposition

$$X_t = A_t + M_t,$$

where A is of local finite variation and M is a local martingale (see Appendix A.3). The most common special case of this is of course when X is an Ito process and thus has dynamics of the form

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where W is Wiener.

4.1 Definition, existence, and basic properties

It would seem natural to define occupation time $T_t(A)$ in the usual way as $T_t(A) = \int_0^t I_A(X_s) ds$ and then define local time as $L_t^x = T_t(dx)/dm$. It is perfectly possible to develop a theory based on these definitions, but instead it has become standard to measure occupation time with respect to the quadratic variation process $\langle X \rangle$. The concept of quadratic variation belongs to general semimartingale theory, but for Ito processes it is very simple. See Appendix A.1 for details.

Definition 4.1 *For an Ito process with dynamics*

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

we define the quadratic variation process $\langle X \rangle$ by

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds,$$

i.e.

$$d\langle X \rangle_t = \sigma_t^2 dt, \quad \text{or} \quad d\langle X \rangle_t = (dX_t)^2.$$

We can now define occupation time.

Definition 4.2 *For any Borel set $A \subseteq \mathbb{R}$ we define occupation time $T_t(A)$ by*

$$T_t(A) = \int_0^t I_A(X_s) d\langle X \rangle_s,$$

so, for an Ito process we have

$$T_t(A) = \int_0^t I_A(X_s) \sigma_s^2 ds.$$

We note that for a standard Wiener process W , we have $d\langle W, W \rangle_t = dt$, so for a Wiener process the new definition coincides with the old one. Local time is now defined as before.

Definition 4.3 *If $T_t(dx) \ll dm$ then we define **local time** L_t^x by*

$$L_t^x = \frac{T_t(dx)}{m(dx)}.$$

With the new definition of occupation time we also see that that if $\sigma_t \equiv 0$ (so X is of bounded variation), then $T_t \equiv 0$. Thus local time does (trivially) exist and we have $L \equiv 0$.

The interesting case is thus when $\sigma \neq 0$. We have the following result, where the first item is deep and the other items are more or less expected.

Theorem 4.1 *Assume that X is a continuous semimartingale with canonical decomposition $X = A + M$. Then the following hold.*

- *Local time L_t^x exists. Furthermore, there is a version of L which is continuous in t and cadlag in x .*
- *The jump size of L is given by*

$$L_t^x - L_t^{x-} = 2 \int_0^t I\{X_s = x\} dA_s.$$

- *If X is a local martingale, then L is almost surely continuous in (t, x) .*
- *For every fixed x , process L_t^x is nondecreasing and only increases on the set $\{X_t = x\}$.*
- *For every Borel set A we have*

$$\int_0^t I_A(X_s) d\langle X \rangle_s = \int_{\mathbb{R}} I_A(x) L_t^x dx$$

- *For every bounded measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ we have*

$$\int_0^t f(X_s) d\langle X \rangle_s = \int_{\mathbb{R}} f(x) L_t^x dx.$$

- *We have the informal interpretation*

$$L_t^x = \int_0^t \delta_x(X_s) d\langle X \rangle_s.$$

Note that a discontinuity in $x \mapsto L_t^x$ can only occur if the finite variation process A charges the set $\{X_s = x\}$. A concrete example is given in Section 4.5.

4.2 A random time change

We now prove that local time is preserved under a random time change (see Appendix B for definitions and details on random time changes). Let us therefore consider a semimartingale X , and a smooth random change of time $t \mapsto C_t$. We define the process X^C by

$$X_t^C = X_{C_t},$$

and we denote the local time of X and X^C by $L(X)$ and $L(X^C)$ respectively.

Proposition 4.1 *With notation as above we have*

$$L_t^a(X^C) = L_{C_t}^a(X).$$

Proof. By definition we have

$$L_t^a(X) = \int_0^t \delta_a(X_s) d\langle X \rangle_s,$$

so we have

$$L_{C_t}^a(X) = \int_0^{C_t} \delta_a(X_s) d\langle X \rangle_s,$$

and thus

$$dL_{C_t}^a(X) = \delta_a(X_{C_t}) d\langle X \rangle_t^C$$

On the other hand, we have by definition

$$L_t^a(X^C) = \int_0^t \delta_a(X_s^C) d\langle X^C \rangle_s,$$

so

$$dL_t^a(X^C) = \delta_a(X_{C_t}) d\langle X^C \rangle_t.$$

From Proposition B.1 we have $\langle X^C \rangle_t = \langle X \rangle_{C_t}$ so we have

$$dL_t^a(X^C) = \delta_a(X_{C_t}) d\langle X \rangle_t^C$$

which gives us

$$dL_t^a(X^C) = dL_{C_t}^a(X). \quad \blacksquare$$

4.3 The Tanaka formula

Assume again that X is a semimartingale and consider the process Z defined by

$$Z_t = |X_t|.$$

We now apply the Ito formula exactly like in the case of Brownian motion. Interpreting the derivatives of $|x|$ in the distributional sense we would then formally obtain

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_s) dX_s + \int_0^t \delta_0(X_s) d\langle X \rangle_s$$

and from Theorem 4.1, we conclude that we can write this as

$$|X_t| = |X_0| + \int_0^t \operatorname{sgn}(X_s) dX_s + L_t^0.$$

Applying the same argument to the function $|x - a|$ we obtain, at least informally, the following result.

Theorem 4.2 (Tanaka's formula) *With notation as above we have, for every fixed $a \in R$,*

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a.$$

The Tanaka formula leads us to an alternative definition of local time. This alternative way of defining local time for a diffusion is in fact the most common definition in the literature.

Definition 4.4 *Assume that X is a continuous semimartingale. We may then define local time L_t^a by the formula*

$$L_t^a(X) = |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_s - a) dX_s.$$

4.4 An extended Ito formula

Let us consider a convex function $f : R \rightarrow R$, so f' exists almost everywhere. It furthermore follows from the convexity of f that we may interpret the second derivative f'' as a positive measure $f''(dx)$. In most practical applications this means that we can view f'' as consisting of two parts: an absolutely continuous part, and a finite number of point masses. In most cases we can therefore write $f''(dx)$ as

$$f''(dx) = g(x)dx + \sum_i c_i \delta_{a_i}(x)dx$$

where the density g is the absolutely continuous part of f'' , and the point mass at $x = a_i$ has size c_i .

If X is a semimartingale, a formal application of the Ito formula gives us

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s,$$

and from Theorem 4.1, we conclude that we can write the last term as

$$\int_0^t f''(x)d\langle X \rangle_s = \int_R L_t^a f''(a)da.$$

The fact that we can view f'' as a measure finally allows us to write

$$\int_0^t f''(x)d\langle X \rangle_s = \int_R L_t^a f''(da).$$

We have thus given a heuristic argument for the following extended Ito formula which, among other things, streamlines the argument behind the Tanaka formula.

Theorem 4.3 (Extended Ito formula) *Assume that X is a continuous semimartingale and that f is the difference between two convex functions. Then we have*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2} \int_0^t f''(X_s)d\langle X \rangle_s$$

where f'_- is the left hand derivative (which exists everywhere) of f , and f'' is interpreted in the distributional sense. Alternatively we may write

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2} \int_R L_t^a f''(da).$$

4.5 An example: The local time at zero of $|W|$

To see that a semimartingale may indeed have a non-continuous local time, is instructive to study the local time of the process

$$X_t = |W_t|.$$

Intuitively, and using the reflection principle of Brownian motion, one would guess that

$$L_t^x(X) = 2L_t^x(W), \quad \text{for } x \geq 0$$

whereas

$$L_t^x(X) = 0, \quad \text{for } x < 0.$$

This is in fact correct, and to see this formally, let us in particular study $L_t^0(X)$. From Tanaka we have

$$dX_t = \text{sgn}(W_t)dW_t + dL_t^0(W).$$

Again from the Tanaka formula, and the fact that $X_t = |X_t|$, we have

$$dX_t = \text{sgn}(X_t)dX_t + dL_t^0(X) = [1 - 2I\{X_t = 0\}]dX_t + dL_t^0(X).$$

We thus obtain

$$\begin{aligned} dX_t &= dX_t - 2I\{X_t = 0\} \operatorname{sgn}(W_t) dW_t - 2I\{X_t = 0\} dL_t^0(W) + dL_t^0(X), \\ &= dX_t - 2dL_t^0(W) + dL_t^0(X), \end{aligned}$$

where we have used the facts that $I\{X_t = 0\} \operatorname{sgn}(W_t) dW_t = 0$, and that we have $I\{X_t = 0\} dL_t^0(W) = dL_t^0(W)$. We have thus proved (as expected) that

$$L_t^0(|W|) = 2L_t^0(W).$$

Since we obviously have $L_t^x(X) = 0$ for $x < 0$, we thus see that $L_t x(X)$ has a discontinuity at $x = 0$ of size

$$L_t^0(X) - L_t^{0-}(X) = 2L_t^0(W),$$

and this is exactly what is stated in Theorem 4.1.

4.6 Regulated SDE:s

In this section we will extend the Skorohod construction of regulated Brownian motion in Section 3.5 to the case of a regulated diffusion. The basic idea is the following.

- We consider an SDE of the form

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t, \\ X_0 &= 0 \end{aligned}$$

- We would now like to regulate the process X in such a way that it stays positive, and we want to achieve this with a minimal (in some sense) effort.
- We thus add an increasing process F to the right hand side of the SDE above, such that F only increases when $X_t = 0$.

We start by formally defining the concept of a regulated SDE.

Definition 4.5 *For given functions μ and σ , consider the **regulated SDE***

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t + dF_t, \\ X_0 &= 0. \end{aligned}$$

*We say that a pair of adapted processes (X, F) is a **solution** of the regulated SDE above if the following conditions are satisfied*

- *The pair (X, F) does indeed satisfy the regulated SDE above.*
- *For all t we have $X_t \geq 0$, P -a.s.*
- *The process F is increasing and $F_0 = 0$.*

- F is only increasing on the set $\{X_t = 0\}$.

We now have the following result by Skorohod.

Theorem 4.4 (Skorohod) *Assume that μ and $\sigma \neq 0$ are C^1 , and that they satisfy the Lipschitz conditions*

$$|\mu(x) - \mu(y)| \leq K|x - y|, \quad |\sigma(x) - \sigma(y)| \leq K|x - y|,$$

for some constant K . Then the regulated SDE

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t + dF_t, \\ X_0 &= 0. \end{aligned}$$

has a unique solution (X, F) . We can furthermore identify F with local time at zero of X , so

$$F_t = L_t^0(X).$$

Proof. We divide the proof into several steps.

1. Uniqueness.

Consider two solutions (X, F) and (Y, G) . The difference $Z = X - Y$ will satisfy the equation

$$dZ_t = Z_t \widehat{\mu}_t dt + \widehat{\sigma}_t dW_t + dF_t - dG_t,$$

where

$$\widehat{\mu}_t = \begin{cases} \frac{\mu(X_t) - \mu(Y_t)}{X_t - Y_t} & \text{when } X_t \neq Y_t \\ \mu'(X_t) & \text{when } X_t = Y_t. \end{cases}$$

From Ito we obtain

$$\begin{aligned} dZ_t^2 &= 2Z_t dZ_t + (dZ_t)^2 \\ &= Z_t^2 \{2\widehat{\mu}_t + \widehat{\sigma}_t^2\} dt + 2Z_t^2 \widehat{\sigma}_t dW_t + (X_t - Y_t) \{dF_t - dG_t\}. \end{aligned}$$

We easily see that $(X_t - Y_t) \{dF_t - dG_t\} = -(Y_t dF_t + X_t dG_t) \leq 0$. Furthermore, the Lipschitz conditions on μ and σ imply that $\widehat{\mu}$ and $\widehat{\sigma}$ are bounded by K . Taking expectations and defining h by $h_t = E[Z_t^2]$ we thus obtain

$$\frac{dh}{dt} \leq h_t (2K + K^2), \quad h_0 = 0,$$

so Gronwall's inequality gives us $h \equiv 0$ and thus $Z \equiv 0$.

2. Existence for the case $\mu = 0, \sigma = 1$

For this case have the equation

$$dX_t = dW_t + dF_t,$$

so this case is already covered by Proposition 3.3.

3. Existence for the case with $\mu = 0$ and $\sigma \neq 0$

We now have an equation of the form

$$dX_t = \sigma(X_t)dW_t + dF_t.$$

From step 2 we know that there exists a solution (X^0, F^0) to the equation

$$dX_t^0 = dW_t^0 + dF_t^0.$$

where W^0 is a Wiener process. We now define the random time change $t \mapsto C_t$ by

$$t = \int_0^{C_t} \frac{1}{\sigma^2(X_s)} ds,$$

and define X and F by

$$X_t = X_{C_t}^0, \quad F_t = F_{C_t}^0$$

It now follows from Proposition B.4 that (X, F) is the solution to

$$dX_t = \sigma(X_t)dW_t + F_t,$$

where W is a Wiener process.

4. Existence for the general case $\mu \neq 0$, $\sigma \neq 0$

This case is easily reduced to step 3 by a Girsanov transformation, and we note that the properties of F are not changed under an absolutely continuous measure transformation.

5. Identifying F as local time

In step 2 above we know from Section 3.6 that we have the identification $F_t^0 = L_t^0(X)$. When we perform the time change in step 3 the local time property is preserved due to Proposition 4.1. Finally, the Girsanov transformation in step 4 does not affect this identification (although it will of course change the distribution of L). ■

5 Some formal proofs

In this section we present (at least partial) formal proofs for the main results given above, and for most of the proofs we follow [3]. These proofs require some more advanced techniques and results from stochastic analysis. We start with the definition of local time.

Definition 5.1 *Let X be a semimartingale with canonical decomposition $X = M + A$. The local time L_t^x is defined by*

$$L_t^x = |X_t| - |X_0| - \int_0^t \operatorname{sgn}(X_t - x) dX_t. \quad (1)$$

5.1 Basic properties of L

The first result is surprisingly easy to prove.

Proposition 5.1 *Local time has the following properties*

- For every fixed x , the process $t \mapsto L_t^x$ is non negative, continuous, and nondecreasing.
- For every fixed x , the process L^x is supported by the set $\{X_t = x\}$ in the sense that

$$\int_0^\infty I\{X_t \neq x\} dL_t^x = 0.$$

- We have the representation

$$L_t^x = -\inf_{s \leq t} \int_0^s \operatorname{sgn}(X_u - x) dX_u$$

Proof. We restrict ourselves to the case when X is a local martingale. It is enough to do the proof for the case $x = 0$. For each $\epsilon > 0$ we can find a convex C^2 -function f such that $f_h(x) = |x|$ for $x \leq 0$ and $f(x) = x - h$ for $x \geq h$. We note that

$$\begin{aligned} f_h''(x) &= 0, & \text{for } x \leq 0 \text{ and } x \geq h, \\ f_h(x) &\rightarrow |x|, & \text{as } h \rightarrow 0, \\ f_h'(x) &\rightarrow \operatorname{sgn}(x), & \text{as } h \rightarrow 0, \end{aligned}$$

We now define the process Z_t^h by

$$Z_t^h = f_h(X_t) - \int_0^t f_h'(X_s) dX_s.$$

Since $f_h(x)$ is a good approximation of $|x|$, we expect that Z_t^h should be a good approximation to L_t^0 . We obviously have $f_h(X_t) \rightarrow |X_t|$. Furthermore, since $f_h'(X_t) \rightarrow \operatorname{sgn}(X_t)$ and $|f_h'(X_t)| \leq 1$ we conclude from the stochastic dominated convergence Theorem A.3 that

$$(Z^h - L^0)^* \xrightarrow{P} 0$$

where for any process Y we define Y^* by $Y_t^* = \sup_{s \leq t} |Y_s|$.

We have thus seen that Z_t^h converges uniformly in probability to L_t^0 . We now apply the Ito formula to Z_t^h and obtain

$$Z_t^h = \int_0^t f_h''(X_s) d\langle X \rangle_s.$$

Since f_h is convex and $\langle X \rangle$ is nondecreasing we see that Z_t^h is increasing, and since Z_t^h converges to L_t^0 we conclude that L_t^0 is nondecreasing. It is also obvious from the definition that L_t^0 is continuous. We have thus proved the first statement in the proposition.

To show the second statement we note that since $f_h'' = 0$ outside $[0, h]$ we trivially have

$$\int_0^t I\{X_s \notin [0, h]\} dZ_s^h = \int_0^t I\{X_s \notin [0, h]\} f_h''(X_s) d\langle X \rangle_s = 0$$

Going to the limit this gives us

$$\int_0^t I\{X_s \neq 0\} dL_s^0 = 0.$$

The third statement follows from the Skorohod result. ■

We now go on to prove continuity in (t, x) of L_t^x . The obvious idea is to use the Kolmogorov criterion (see Theorem C.1) for continuity, and this will require the Burkholder-Davis-Gundy inequality from Theorem A.1.

Proposition 5.2 (Continuity of L_t^x) *With L_t^x defined by Definition 5.1 the following hold.*

- *There is a version of L which is continuous in t and cadlag in x .*
- *The jump size of L is given by*

$$L_t^x - L_t^{x-} = 2 \int_0^t I\{X_s = x\} dA_s. \quad (2)$$

- *If X is a local martingale, then L is almost surely continuous in (t, x) .*

Proof. By definition and with $M_t = \int_0^t \sigma_s dW_s$, we have

$$L_t^x = |X_t - x| - |X_0 - x| - \int_0^t \operatorname{sgn}(X_s - x) dM_s - \int_0^t \operatorname{sgn}(X_s - x) dA_s$$

The term $|X_t - x|$ is obviously continuous in (t, x) and the jump size of the dA -integral is clearly given by the right hand side of (2). It thus remains to prove continuity of the integral term

$$I_t^x = \int_0^t \operatorname{sgn}(X_s - x) dM_s.$$

By localization we may assume that the processes $X_t - X_0$, $\langle M \rangle_t^{\frac{1}{2}}$ and $\int_0^t |dA_s|$ are bounded by some constant C . For any $x < y$ the BDG-inequality (6) gives

us

$$\begin{aligned} E [(I^x - I^y)_t^{*p}] &= 2^p E \left[\left(\sup_{s \leq t} \int_0^s I \{x \leq X_u \leq y\} dM_u \right)^p \right] \\ &\leq KE \left[\left(\int_0^s I \{x \leq X_s \leq y\} d\langle M \rangle_s \right)^{p/2} \right], \end{aligned}$$

where we use K to denote any constant.

In order to estimate the last integral we now set $h = y - x$ and choose a C^2 -function f with $f'' \geq 2I_{(x,y]}$, and $|f'| \leq 2h$. Applying the Ito formula and recalling that $\langle X \rangle = \langle M \rangle$ we obtain

$$\begin{aligned} \int_0^t I_{(x,y]}(X_s) d\langle M \rangle_s &\leq \frac{1}{2} \int_0^t f''(X_s) d\langle M \rangle_s = f(X_t) - f(X_0) - \int_0^t f'(X_s) dX_s \\ &= f(X_t) - f(X_0) - \int_0^t f'(X_s) dA_s - \int_0^t f'(X_s) dM_s \\ &\leq 4Ch + \left| \int_0^t f'(X_s) dM_s \right|. \end{aligned}$$

From the BDG inequality we now have

$$E \left[\left(\sup_{s \leq t} \left| \int_0^s f'(X_s) dM_s \right| \right)^{p/2} \right] \leq E \left[\left(\int_0^t [f'(X_s)]^2 d\langle M \rangle_s \right)^{p/4} \right] \leq (2Ch)^{p/2}.$$

We thus have

$$E \left[\left(\sup_{s \leq t} [I_s^x - I_s^y] \right)^p \right] \leq K (Ch)^{p/2}.$$

Choosing any $p > 2$ and using the Kolmogorov criterion in Theorem C.1 gives us the continuity result. ■

5.2 The Tanaka formulas

Given our definition of local time, the Tanaka formulas follow immediately.

Proposition 5.3 (Tanaka's formulas) *For any continuous semimartingale X we have the following relations.*

$$\begin{aligned} |X_t - x| &= |X_0 - x| + \int_0^t \operatorname{sgn}(X_s - x) dX_s + L_t^x(X), \\ (X_t - x)^+ &= (X_0 - x)^+ + \int_0^t I(X_s > x) dX_s + \frac{1}{2} L_t^x(X), \\ (X_t - x)^- &= (X_0 - x)^- - \int_0^t I(X_s \leq x) dX_s + \frac{1}{2} L_t^x(X), \end{aligned}$$

Proof. The first formula is nothing more than Definition 5.1. The second formula follows immediately from applying the first formula to the relation $|x| + x = 2x^+$. The third formula follows in the same fashion. ■

5.3 Local time as occupation time density

Let $f : R \rightarrow R$ be a convex function. A well known result then says that the left hand derivative $f'_-(x)$ exists for every x . One can furthermore show that f'_- is left continuous and nondecreasing, so we can define a Borel measure μ_f on the real line by the prescription

$$\mu_f([x, y)) = f'_-(y) - f'_-(x).$$

We will also denote this measure by $f''(dx)$, and when $f \in C^2$ we will have $\mu_f(dx) = f''(dx) = f''(x)dx$. We now have the following generalization of the Ito and Tanaka formulas.

Theorem 5.1 (Extended Ito-Tanaka formula) *Assume that f is the difference between two convex functions, and that X is a semimartingale with local time L_t^x . Then we have*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2} \int_R L_t^x \mu_f(dx). \quad (3)$$

Proof. When $f(x) = |x|$ this is just the Tanaka formula with μ_f being a point mass of size 2 at $x = 0$. If μ_f has finite support, so μ_f consists of a finite number of point masses, corresponding to an f with a finite number of discontinuities in f' , the formula follows easily from the Tanaka formulas and linearity. The general case can then be proved by approximating a general μ_f with a sequence of measures with finite support. See [3] for details. ■

We now have the following consequence of the extended Ito-Tanaka formula.

Theorem 5.2 (Occupation time density) *For every nonnegative Borel function $f : R \rightarrow R_+$ we have*

$$\int_0^t f(X_s)d\langle X \rangle_s = \int_R f(x)L_t^x dx. \quad (4)$$

Proof. Suppose that $f \in C$. Then we may interpret f as $f = F''$ for a convex function $F \in C^2$. Applying formula (3) to $F(X_t)$ and comparing the result to the standard Ito formula, shows that (4) holds in this case. The general case of a nonnegative Borel function f then follows by a standard monotone class argument. ■

6 Notes to the literature

Full proofs of the main results (and much more) can be found in many textbooks on stochastic calculus, such as [2], [3], [4], [7], and [8]. A very readable old classic is [5] which also contains results on regulated diffusions. Apart from a treatment of the modern approach, [4] also contains an exposition of local time based on the original Levy theory of excursions. An approach based on random walk arguments is presented in [6]. The handbook [1] contains a wealth of results.

A Stochastic Calculus for Continuous Martingales

In this appendix we provide some basic concepts and facts concerning stochastic integrals and the Ito formula for continuous semimartingales. We consider a given filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F})$, where $\mathbf{F} = \{\mathcal{F}_t\}_t \geq 0$. Now consider a process X and a process property E . E could be anything, like being a martingale, being square integrable, being bounded, having finite variation etc.

Definition A.1 *We say that X has the property E locally if there exists an increasing sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} \tau_n = \infty$ such that the stopped process X^{τ_n} defined by*

$$X_t^{\tau_n} = X_{t \wedge \tau_n}$$

has the property E for each n .

We will sometimes use the term *predictable* in the text below. For our purposes it is enough to know that any adapted process with left continuous trajectories is predictable.

A.1 Quadratic variation

Suppose that X is a locally square integrable martingale with continuous trajectories. Then X^2 will be a submartingale, and from the Doob-Meyer Theorem we know that there exists a unique predictable increasing process A with $A_0 = 0$ such that the process

$$X_t^2 - A_t$$

is a martingale, and one can show that A is in fact continuous. The process A above is very important for stochastic integration theory so we give it a name.

Definition A.2 *For any locally square integrable martingale X , the **quadratic variation process** $\langle X \rangle$ is defined by*

$$\langle X \rangle_t = A_t,$$

where A is defined above.

If X is a Wiener process W , then it is easy to see that $W_t - t$ is a martingale, so we have the following trivial result.

Lemma A.1 *For a Wiener process W , we have $\langle W, W \rangle_t = t$*

The name “quadratic variation” is motivated by the following result

Proposition A.1 *Let X be as above, consider a fixed $t > 0$ and let $\{p^n\}_{n=1}^\infty$ be a sequence of partitions of the interval $[0, t]$ such that the mesh (the longest subinterval) of p^n tends to zero as $n \rightarrow \infty$. Define S_t^n by*

$$S_t^n = \sum_i [X(t_{i+1}^n) - X(t_i^n)]^2$$

Then S_t^n converges in probability to $\langle X \rangle_t$.

This results motivates the formal expression

$$d\langle X \rangle_t = (dX_t)^2, \tag{5}$$

and we note that for the case when X is a Wiener process W , this is the usual “multiplication rule” $(dW_t)^2 = dt$.

We end this section by quoting the Burkholder-Davis-Gundy (BDG) inequality. This inequality shows how the maximum of a local martingale is controlled by its quadratic variation process.

Theorem A.1 (Burkholder-Davis-Gundy) *For every $p > 0$ there exists a constant C_p such that*

$$E [(M_t^*)^p] \leq E [\langle M \rangle_t^{p/2}] \tag{6}$$

for every continuous local martingale M , where M^ is defined by*

$$M_t^* = \sup_{s \leq t} |M_s|.$$

A.2 Stochastic integrals

Assume that M is a continuous square integrable martingale one can quite easily define the stochastic integral

$$\int_0^t g_s dM_s$$

almost exactly along the lines of the construction of the usual Ito integral. The basic properties of the integral are as follows.

Proposition A.2 *Let M be a square integrable martingale, and let g be an adapted process satisfying the integrability condition*

$$E \left[\int_0^t g_s^2 d\langle M \rangle_s \right] < \infty$$

for all t . Then the integral process $g * M$ defined by

$$(g * M)_t = \int_0^t g_s dM_s$$

is well defined and has the following properties.

- The process $g * M$ is a square integrable martingale with continuous trajectories.

- We have

$$E \left[\left(\int_0^t g_s dM_s \right)^2 \right] = E \left[\int_0^t g_s^2 d\langle M \rangle_s \right]$$

- We have

$$d\langle g * M \rangle_t = g_t^2 d\langle M \rangle_t. \quad (7)$$

The proof of the first three items above are very similar to the standard Ito case. The formula (7) follows from (5) and the fact that $d(g * M)_t = g_t dM_t$.

The stochastic integral above can quite easily be extended to a larger class of processes. Note that a continuous local martingale is also locally square integrable.

Definition A.3 For a continuous local martingale M , we define $L^2(M)$ as the class of adapted processes g such that

$$\int_0^t g_s^2 d\langle M \rangle_s < \infty, \quad P - a.s.$$

for all $t \geq 0$.

We now have the following result.

Proposition A.3 Let M be a local martingale, and let g be an adapted process in $L^2(M)$. Then the integral process $g * M$ defined by

$$(g * M)_t = \int_0^t g_s dM_s$$

is well defined and has the following properties.

- The process $g * M$ is a local martingale with continuous trajectories.

- We have

$$d\langle g * M \rangle_t = g_t^2 d\langle M \rangle_t. \quad (8)$$

A.3 Semimartingales and the Ito formula

We start by defining the semimartingale concept.

Definition A.4 *If a process X has the form*

$$X_T = A_t + M_t,$$

*where A is adapted with $A_0 = 0$, continuous and of (locally) bounded variation, and M is a local martingale, then we say that X is a **semimartingale**. If A can be written on the form*

$$dA_t = \mu_t dt,$$

*then we say that X is a **special semimartingale**.*

We have the following uniqueness result.

Lemma A.2 *The decomposition $X = M + A$ is unique.*

Given the stochastic integral defined earlier, we see that semimartingales are natural integrators.

Definition A.5 *Let X be a semimartingale as above. The class $L(X)$ is defined as*

$$L(X) = L(M) \cap L^1(|dA|).$$

where $L^1(|dA|)$ is the class of adapted processes g such that

$$\int_0^t |g_s| |dA|_s < \infty.$$

For $g \in L(X)$ we define the stochastic integral by

$$\int_0^t g_s dX_s = \int_0^t g_s dM_s + \int_0^t g_s dA_s.$$

The quadratic variation of X is defined by

$$d\langle X \rangle_t = d\langle M \rangle_t.$$

It is easy to show that any function of bounded variation has zero quadratic variation, so (5) can be extended to

$$d\langle X \rangle_t = (dX_t)^2.$$

Very much along the lines of the informal rules $(dt)^2 = 0$ and $dt \cdot dW_t = 0$ in standard Ito calculus we can thus motivate the informal multiplication table

$$\begin{aligned} (dX_t)^2 &= d\langle X \rangle_t, \\ (dA_t)^2 &= 0, \\ dA_t \cdot dM_t &= 0. \end{aligned}$$

Let us now consider a semimartingale X as above, and assume that $f : R \rightarrow R$ is a smooth function. Inspired by the standard Ito formula we would then intuitively write

$$dF(X_t) = F'(X_t)dX_t + \frac{1}{2}F''(X_t)(dX_t)^2.$$

Here we interpret the dX integral as

$$F'(X_t)dX_t = F'(X_t)dA_t + F'(X_t)dM_t,$$

where the first integral is a Lebesgue-Stieltjes integral (for each ω), whereas the second integral is a stochastic integral. Given the multiplication table above, we also have

$$(dX_t)^2 = (dA_t + dM_t)^2 = (dM_t)^2,$$

so from (5) we would finally expect to obtain

$$(dX_t)^2 = d\langle M \rangle_t$$

This intuitive argument can in fact be made precise and we have the following extension of the Ito formula.

Theorem A.2 (The Ito Formula) *Assume that X is a semimartingale and that $F(t, x)$ is $C^{1,2}$. We then have the Ito formula*

$$dF(t, X_t) = \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(t, X_t)d\langle X \rangle_t.$$

We end the section by quoting the extremely useful dominated stochastic convergence theorem.

Theorem A.3 (Dominated convergence) *Let X be a continuous semimartingale and let Y, Z, Y^1, Y^2, \dots be processes in $L(X)$. Assume that the following conditions hold for all t .*

$$\begin{aligned} Y_t^n &\rightarrow Y_t, & P - a.s. \\ |Y_t^n| &\leq |Z_t|. \end{aligned}$$

Then we have

$$\left\{ \sup_{s \leq t} \left| \int_0^s Y_u^n dX_u - \int_0^s Y_u dX_u \right| \right\} \xrightarrow{P} 0.$$

A.4 The Levy characterization of Brownian motion

Let W be a Wiener process. It is very easy to see that W has the following properties

- W is a continuous martingale.

- The process $W_t^2 - t$ is a martingale.

Surprisingly enough, these properties completely characterize the Wiener process. We have the following result.

Proposition A.4 (Levy) *Assume that the process X has the following properties.*

- X has continuous trajectories and $X_0 = 0$.
- The process X_t is a martingale.
- The process $X_t^2 - t$ is a martingale.

Then X is a standard Wiener process.

We have the following easy corollary.

Corollary A.1 *Assume that X is a continuous martingale with quadratic variation given by*

$$d\langle X \rangle_t = dt.$$

Then X is a standard Wiener process.

Proof. We apply the Ito formula to X^2 to obtain

$$d(X_t)^2 = 2X_t dX_t + d\langle X \rangle_t = 2X_t dX_t + dt,$$

and, since X is a martingale, the term $X_t dX_t$ is a martingale increment. ■

We also have the following easy and useful consequence of the Levy characterization.

Proposition A.5 *Assume that X is a continuous martingale such that*

$$d\langle X \rangle_t = \sigma_t^2 dt,$$

for some process $\sigma > 0$. Then the process W , defined by

$$W_t = \frac{1}{\sigma_s} dX_s$$

is a standard Wiener process, so we can write

$$dX_t = \sigma_t dW_t,$$

where W is standard Wiener.

Proof. Since W is a stochastic integral with respect to a martingale we conclude that W is a martingale. Using (7) we obtain

$$d\langle W, W \rangle_t = \frac{1}{\sigma_t^2} d\langle X \rangle_t = dt. \quad \blacksquare$$

B A random time change

In the section we will investigate how the structure of a semimartingale changes when we perform a random time change.

Definition B.1 A **random time change** is a process $\{C_t : t \geq 0\}$ such that

- $C_0 = 0$.
- C has strictly increasing continuous trajectories.
- For every fixed $t \geq 0$, the random time C_t is a stopping time.

We say that C is **smooth** if the trajectories are differentiable.

In a more general theory of random time changes, we only require that C is non decreasing, but the assumption of strictly increasing continuous trajectories makes life much easier for us.

Definition B.2 If X is an adapted process and C is a random time change, the process X^C is defined as

$$X_t^C = X_{C_t}.$$

We remark that if X is continuous and adapted to the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ then X^C is adapted to the filtration $\mathbf{F}^C = \{\mathcal{F}_{C_t}\}_{t \geq 0}$.

The following extremely useful result tells us how the quadratic variation changes after a time change.

Proposition B.1 Let X be a local \mathbf{F} -martingale and let C be a continuous random time change. Then the following hold.

- X^C is a local \mathbf{F}^C -martingale.
- The quadratic variation of X^C is given by

$$\langle X^C, X^C \rangle_t = \langle X \rangle_t^C.$$

Proof. The first part follows from the optional sampling theorem. For the second part we assume WLOG that X is a square integrable martingale and recall that $A_t = \langle X \rangle_t$ is the unique continuous process of bounded variation such that

$$X_t^2 - A_t$$

is an \mathbf{F} -martingale. From the optional sampling theorem we conclude that

$$(X^2)_{C_t} - A_{C_t}$$

is an \mathbf{F}^C -martingale. On the other hand we trivially have

$$A_{C_t} = A_t^C \quad \text{and} \quad (X^2)_{C_t} = (X^C)_t^2,$$

so

$$(X^C)_t^2 - A_t^C$$

is an \mathbf{F}^C -martingale. ■

We will now investigate the effect of a random time change to a stochastic integral. Let us consider a local martingale X of the form

$$X_t = \int_0^t \sigma_s dW_s,$$

and a random time change C_t . We know that

$$\langle X \rangle_t = \int_0^t \sigma_s^2 ds,$$

so from Proposition B.1 we see that

$$\langle X^C \rangle_t = \int_0^{C_t} \sigma_s^2 ds, \tag{9}$$

so, if the time change is smooth, we have

$$d\langle X^C \rangle_t = \sigma_{C_t}^2 C_t' dt \tag{10}$$

From this we immediately have the following important which follows directly from (10) and Proposition A.5.

Proposition B.2 *Let X be defined by*

$$X_t = \int_0^t \sigma_s dW_s,$$

and consider a smooth time change C_t . Then we can write

$$dX_t^C = \sigma_{C_t} \sqrt{C_t'} d\tilde{W}_t,$$

where \tilde{W} is standard Wiener.

The most commonly used special cases are the following.

Proposition B.3 *With X as above, define the random time change C by the implicit relation*

$$\int_0^{C_t} \sigma_s^2 ds = t,$$

i.e.

$$C_t = \inf \left\{ s \geq 0 : \int_0^s \sigma_u^2 du = t \right\}$$

Then the process $X_t^C = X_{C_t}$ is a standard Wiener process.

Proof. Direct differentiation shows that

$$C'_t = \frac{1}{\sigma_{C_t}^2},$$

and we can now use Proposition B.2. ■

Proposition B.4 *Assume that \tilde{W} is standard Wiener, define the random time change C by the implicit relation*

$$\int_0^{C_t} \sigma_s^{-2} ds = t,$$

i.e.

$$C_t = \inf \left\{ s \geq 0 : \int_0^s \sigma_u^{-2} du = t \right\}$$

Then the process $\tilde{W}_t^C = \tilde{W}_{C_t}$ can be written as

$$\tilde{W}_t^C = \int_0^t \sigma_s dW_s$$

where W is standard Wiener.

We may in fact push this analysis a bit further. Let us therefore consider an SDE of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

and let us define the time change C by

$$\int_0^{C_t} g_s^2(X_s) ds = t.$$

for some deterministic function $g > 0$. This transformation is in fact smooth and we have

$$C'_t = \frac{1}{g^2(X_t^C)}$$

We have

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s$$

so we obtain

$$X_t^C = X_0 + \int_0^{C_t} \mu(X_s) ds + \int_0^{C_t} \sigma(X_s) dW_s.$$

Defining Y by

$$Y_t = \int_0^{C_t} \mu(X_s) ds$$

we obtain

$$dY_t = \mu(X_t^C) C'_t dt = \frac{\mu(X_t^C)}{g^2(X_t^C)} dt$$

Defining Z by

$$Z_t = \int_0^{C_t} \sigma(X_s) dW_s$$

and using Proposition B.1 we obtain

$$\langle Z, Z \rangle_t = \int_0^{C_t} \sigma^2(X_s) ds$$

so on differential form we have

$$d\langle Z, Z \rangle_t = \sigma^2(X_t^C) C'_t dt = \frac{\sigma^2(X_t^C)}{g^2(X_t^C)} dt.$$

It now follows from Proposition A.5 that we can write

$$dZ_t = \frac{\sigma(X_t^C)}{g(X_t^C)} d\tilde{W}_t,$$

where \tilde{W} is standard Wiener. We have thus proved the following result.

Proposition B.5 *Consider the SDE*

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t.$$

Consider furthermore a function $g > 0$ and a random time change of the form

$$\int_0^{C_t} g^2(X_s) ds = t,$$

or equivalently

$$C_t = \inf \left\{ s \geq 0 : \int_0^s g^2(X_u) du = t \right\}.$$

Defining Y by $Y = X^C$, the process Y will satisfy the SDE

$$dY_t = \mu_Y(Y_t) dt + \sigma_Y(Y_t) d\tilde{W}_t,$$

where

$$\begin{aligned} \mu_Y(y) &= \frac{\mu(y)}{g^2(y)}, \\ \sigma_Y(y) &= \frac{\sigma(y)}{g(y)} \end{aligned}$$

and \tilde{W} is standard Wiener.

C The Kolmogorov continuity criterion

Whenever you want to prove continuity for some (possibly multi-indexed) process, the Kolmogorov criterion is often the first choice.

Theorem C.1 (Kolmogorov continuity criterion) *Let $X : \Omega \times \mathbb{R}^d \rightarrow S$ be a process indexed by \mathbb{R}^d and taking values in some complete metric space (S, ρ) . Assume that for some $a, b, C > 0$ we have*

$$E [\{\rho(X_s, X_t)\}^a] \leq C \|s - t\|^{d+b} \quad \text{for all } s, t \in \mathbb{R}^d.$$

Then X has a continuous version.

The two most common choices of S is $S = \mathbb{R}^n$ or $S = C(R)$ with the uniform topology.

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